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A THEORY OF SELF-FOCUSING IN A MAGNETIZED PLASMA

by

(C) PHILIP LAWRENCE PITT

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ABSTRACT

The propagation of intense radiation in a magnetized plasma is investigated theoretically. The presence of intense radiation in a plasma can alter the plasma density profile in the region of the beam. This alteration can be caused by the ponderomotive force or by the heating of the plasma by the radiation. We have calculated the equilibrium density profiles and hence the dielectric ϵ, for each mechanism. The wave equation was then solved using a WKB-method and differential equations were derived that govern the propagation of the beam in the plasma. It is shown that the behaviour of the beam propagation is determined by a competition between diffraction of the beam and the focusing tendency of the beam produced by the altered density profile. The nature of the solution to the wave equation is discussed along with applications to the laser solenoid fusion concept.

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The propagation of intense radiation in a magnetized plasma is investigated theoretically. The presence of intense radiation in a plasma can alter the plasma density profile in the region of the beam. This alteration can be caused by the ponderomotive force or by the heating of the plasma by the radiation. We have calculated the equilibrium density profiles for these two distinct mechanisms. The wave equation was then solved using a WKB method and differential equations were derived that govern the propagation of the beam in the plasma. It is shown that the behaviour of the beam propagation is characterized by a competition between diffraction of the beam and the focusing tendency of the beam produced by the altered density profile. The nature of the solution to the wave equation is discussed along with applications to the laser solenoid fusion concept.

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1. INTRODUCTION

1.1 THE LASER SOLENOID APPROACH TO FUSION

Studies of heating of magnetically confined plasmas to thermonuclear temperatures have led to the investigation of several possible heating schemes, one of which is the laser - solenoid approach proposed by Dawson et al (1971). This scheme consists of a long slender plasma column confined by a solenoidal magnetic field (100 - 500 kG) and intense, long wavelength laser radiation (ie. 10.6μ CO_2 laser) directed axially so as to heat the plasma column to thermonuclear temperatures ($\sim 10^8$ °K). This simple geometry has the advantage of being relatively stable and the heating source (CO_2 laser) holds considerable promise in being scaled up to the required output. However, the physics of the laser plasma interaction must be carefully analysed both theoretically and experimentally before large scale solenoids can be constructed. In this thesis, we consider theoretically one aspect of this interaction; the beam trapping or self - focusing problem.

1.2 REVIEW OF PREVIOUS WORK ON BEAM TRAPPING AND SELF - FOCUSING

Steinhauer and Ahlstrom (1971) showed that a laser

propagating in a theta pinch would be refracted out of the plasma (ie. become untrapped) after propagating a distance of a few beam widths because of the density maximum on axis produced by the pinch. As light tends to bend in the direction of increasing refractive index N, and since N depends on the plasma density n , by $N=(1-n/n^*)$ (where n^* is the critical density for the laser radiation), then if the density had a minimum on axis, the laser light would tend to bend (refract) towards the axis of the solenoid instead of refracting out of the plasma. Using a parabolic density profile, Steinhauer and Ahlstrom (1971) employed ray optics to show that the rays became trapped in the plasma and oscillated about the axis as they propagated along the plasma column.

Humphries (1974) analytically solved for the waveguide type modes in a plasma with a parabolic density profile and determined the width of the smallest fundamental mode. Mani et al (1975) solved the wave equation with a parabolic density profile and a Gaussian intensity profile and produced solutions which showed the beam alternately focusing and defocusing in agreement with ray optics.

In all the above analyses, the plasma was assumed to possess an arbitrary density profile, however it is known that the presence of an intense laser beam can produce a density minimum on axis either by the ponderomotive force (Hora(1971)) or by the heating of the plasma (see Burnett

and Offenberger (1974) for example). Thus the effect of the laser on the plasma can in turn affect the laser beam itself (by causing focusing) so that a self - consistent treatment is necessary. Self - focusing is the result of the laser producing its own favourable density profile either by the ponderomotive force or by heating. The first self - consistent treatment of the ponderomotive mechanism in an unmagnetized plasma was given by Sodha et al (1974) and Max (1976) where oscillatory solutions were found. The heating mechanism was investigated analytically by Sodha et al (1974b) for an unmagnetized plasma and numerically by Feit and Fleck (1976) for a magnetized plasma. Both of these investigations showed the alternate focusing - defocusing behaviour of the beam propagation.

Several experiments have observed beam trapping and they are listed in the references. However, they only demonstrate this effect over short distances and for temperatures much less than the thermonuclear regime.

1.3 NATURE OF THE PRESENT WORK

In this thesis we extend the results of Sodha (1974b) and Max (1976) by including a magnetic field inside the plasma. The procedure used is to compute the steady state dielectric of the magnetized plasma (Chapter 2) and then solve the wave equation (Chapter 3) using the WKB method of

Akmanhov et al (1966) and obtain differential equations which describe the focusing and defocusing of the laser beam in the plasma. We then investigate the nature of the solutions to these equations in Chapter 4 and discuss these results and their ramifications for the laser - solenoid fusion scheme in Chapter 5.

We consider in this thesis, two mechanisms that will produce the favourable density profile for focusing. These are; the ponderomotive mechanism and the heating mechanism. In section 2.1 we compute the steady state dielectric ϵ due to the ponderomotive force. Since ion motion is involved (ie. plasma is pushed radially outwards from the laser beam), pulse times are required to be larger than the acoustic transit time $\tau_{ac} = a/c_s$ in order to establish the eqilibrium (where c is the ion sound velocity and a is the 1/e beam width).

In section 2.2 we extend the work of Sodha et al (1974b) and use the kinetic formalism of Sharkrovsky et al (1966) to compute the dielectric ϵ for the case where heating causes a redistribution of the plasma. As the results of this theory turn out to be quite restricted, especially to low intensities, we will denote these results as "weak heating" results. The weak heating results are extended in the next section so as to include the effects of ion heating and thermal conduction where we use a simplified fluid theory approach to compute ϵ. These results will be

denoted as "strong heating" results to differentiate them from the results of section 2.2. In both the heating cases, the heating time, $\tau_h = \frac{3nT}{k_\nu I}$ (where T is in eV, k_ν is the absorption coefficient and I is the laser intensity in Watts/cm^2) must be larger than the acoustic transit time in order to achieve the desired equilibrium.

To simplify the analysis we treat the laser radiation as being linearly polarized which is valid for $\omega^2 >> \Omega_e^2$ (where ω is the laser frequency and Ω_e is the electron cyclotron frequency), as well as consider the radiation to be monochromatic. We also neglect the axial plasma dynamics and thereby consider only the radial equilibria, which means that our treatment is not fully self - consistent.

Cgs units are used throughout except we adhere to the convention of expressing temperature in electron volts (eV) and power in Watts.

2. SELF-CONSISTENT CALCULATION OF THE NON-LINEAR DIELECTRIC IN A MAGNETIZED PLASMA

In this chapter we compute the dielectric ϵ of a magnetized plasma in the presence of strong electromagnetic radiation. We begin in section 2.1 by employing a fluid model for the case of a collisionless plasma where the ponderomotive force is responsible for altering the plasma density profile. In the next two sections we use kinetic and fluid theory respectively to calculate an expression for the dielectric where heating of the plasma by the radiation field alters the density profile.

2.1 COLLISIONLESS PLASMA - PONDEROMOTIVE FORCE

We begin by treating the electrons as a fluid and solve for the motion of the electron fluid in the presence of strong electromagnetic radiation (ie. laser). The equation of motion of the electrons is

$$\{2.1\} \quad \partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{-e}{m_e}\vec{E} - \frac{e}{m_e c}\vec{v}\times\vec{B} - \frac{1}{m_e n_e}\vec{\nabla} p_e$$

where $p_e = n_e T_e$. We solve {2.1} by separating oscillatory and non-oscillatory quanties;

$$\vec{v} = \vec{v}_o + \tilde{v} \quad , \; |\tilde{v}| \ll |\vec{v}_o|$$

$$\vec{\mathcal{E}} = \bar{E}_{e.s} + Re\, \tilde{E}$$

$$\vec{B} = \vec{B}_o + Re\, \tilde{B} \tag{2.2}$$

where $\vec{v}_o$ is the unperturbed fluid velocity and $\tilde{v}$ is the time varying perturbation induced by the radiation field. The total electric field $\vec{\mathcal{E}}$ is composed of the time varying laser field $\tilde{E}$ and an electrostatic field $\bar{E}_{e.s}$ due to any charge separation between electrons and ions. It is assumed that no external d.c. electric field is applied. The total magnetic field $\vec{B}$ is composed of the laser field contribution $\tilde{B}$ and a constant, externally applied d.c. field, $\vec{B}_o=(0,0,B_o)$. All oscillatory quantities are taken to vary as $e^{i\omega t}$, where ω is the frequency of the radiation field, so that substitution of {2.2} into {2.1} yields the zeroth order velocity (for $\omega^2 \gg \Omega_e^2 = e^2B^2/m_e^2c^2$, and $E \gg T/eL$ where L is a typical scale length),

$$\tilde{v} = -e\tilde{E}/i m_e \omega \tag{2.3}$$

We restrict discussion in this section to the collisionless time scale, so that if no collisionless heating takes place, we have

$$\vec{\nabla} p_e = T_e \vec{\nabla} n_e \tag{2.4}$$

Using {2.3} and {2.4} together with the Maxwell equation

$$\tilde{B} = -\frac{c}{i\omega} \tilde{\nabla} \times \tilde{E} \tag{2.5}$$

in {2.1}, we obtain after averaging over fast time scales

$$\frac{\vec{\nabla} n_e}{n_e} = -\frac{e\bar{E}_{e.s.}}{T_e} - \frac{e^2}{2m_e\omega^2 T_e}\left[(\tilde{E}\cdot\nabla)\tilde{E}^* + \tilde{E}\times\tilde{\nabla}\times\tilde{E}^*\right] - \frac{e\,\vec{v}_o\times\vec{B}_o}{m_e c T_e} \tag{2.6}$$

The second term in {2.6} represents the non-linear (ponderomotive) force due to the oscillatory field. It is to be understood that the density n_e in the above equation is the time averaged density $\langle n_e \rangle$. Using Ampere's law

$$\vec{\nabla} \times \vec{B}_o = \frac{4\pi n_e e}{c} \vec{v}_o \tag{2.7}$$

{2.6} becomes,

$$\{2.8\}\quad \frac{\vec{\nabla} n_e}{n_e} = \frac{-e\overline{E}_{e.s.}}{T_e} - \frac{e^2}{2m_e\omega^2 T_e}\left[(\tilde{E}\cdot\vec{\nabla})\tilde{E}^* + \tilde{E}\times\vec{\nabla}\times\tilde{E}^*\right] - \frac{(\vec{\nabla}\times\vec{B}_o)\times\vec{B}_o}{4\pi T_e}$$

Upon using the identities

$$\{2.9\}\qquad \tilde{E}\times\vec{\nabla}\times\tilde{E}^* + (\tilde{E}\cdot\vec{\nabla})\tilde{E}^* = \vec{\nabla}\frac{E^2}{2}$$

$$\{2.10\}\qquad (\vec{\nabla}\times\vec{B}_o)\times\vec{B}_o = (\vec{B}_o\cdot\vec{\nabla})\vec{B}_o - \frac{B_o^2}{2}$$

in {2.8}, we obtain

$$\{2.11\}\qquad \frac{\nabla n_e}{n_e} = -\frac{e\overline{E}_{e.s.}}{T_e} - \frac{e^2}{4m_e\omega^2}\vec{\nabla}E^2 - \frac{1}{n_e T_e}\vec{\nabla}\frac{B_o^2}{8\pi}$$

where we have assumed that the external magnetic field $\vec{B}_o$ to vary only in the radial direction. Equation {2.11} represents the equilibrium equation satisfied by the electron fluid in the presence of a radiation field. The second term on the right-hand-side (the ponderomotive force) can cause a redistribution of electrons thereby producing radial density and magnetic field gradients. The ions are

coupled to the electron motion through the charge separation field $\vec{E}_{es}$.

If the ponderomotive force and the $\vec{\nabla} B_0^2$ force have scale lengths greater than a Debye length, then we can neglect the electrostatic field E over these scale lengths and write {2.11} as

{2.12} $$\frac{\vec{\nabla} n_e}{n_e} = -\alpha_r \vec{\nabla} E^2 - \frac{1}{n_e T_e} \vec{\nabla} \frac{B_0^2}{8\pi}$$

where $\alpha_r = e^2/4m_e\omega^2 T_e$.

We solve {2.12} in the infinite conductivity limit, ie. assuming frozen magnetic field lines, such that

{2.13} $$\frac{n_e}{n_o} = \frac{B}{B_o}$$

where n_o, B_o are the density and magnetic field far away from the laser beam. The solution of {2.12}, using {2.13}, that satisfies the boundary condition; $n_e \longrightarrow n_o$ as $E \longrightarrow 0$ is

{2.14} $$\ln \eta + \frac{2}{\beta_o}(\eta - 1) = -\alpha_r E^2$$

where

$$\eta = n_e/n_o$$

{2.15} $$\beta_o = \frac{8\pi n_o T_e}{B_o^2}$$

For sufficiently high magnetic fields, $\beta_o \ll 1$, and {2.14} becomes

{2.16} $$\eta \simeq 1 - \alpha_r \beta_o E^2/2$$

Thus in this limit, the equilibrium density profile is parabolic in E as compared to the exponential dependence (Max (1976)) found in the field free case (B_o=0). The axial magnetic field can therefore reduce the plasma depletion due to the ponderomotive force, as expected.

With the knowledge of the equilibrium density profile, η, we can write down the steady state dielectric valid for $\omega^2 \gg \Omega_c^2$, and collisionless time scales as

$$\epsilon = 1 - \frac{\omega_{p-}^2}{\omega^2}\eta = \epsilon_L + \Phi(E) \tag{2.17}$$

where $\omega_{p-}^2 = 4\pi n_o e^2/m_e$

$$\epsilon_L = 1 - \omega_{p-}^2/\omega^2 \tag{2.18}$$

$$\Phi(E) = \frac{\omega_{p-}^2}{\omega^2}\left(1 - \eta(E)\right)$$

The non-linear term, $\Phi(E^2)$, of the dielectric vanishes for $E^2 \longrightarrow 0$ as the plasma becomes a linear medium $(\epsilon \rightarrow \epsilon_L)$ with no strong electromagnetic field present. For the case of the ponderomotive force in a strong magnetic field $(\beta_o << 1)$ Φ becomes

$$\Phi(E) = \frac{\omega_{p-}^2 \beta_o \alpha_r E^2}{2\omega^2}, \quad \beta_o << 1 \tag{2.19}$$

This expression will be used later in the solution of the wave equation and the subsequent investigation of self-

focusing.

2.2 COLLISIONAL HEATING - KINETIC THEORY

In this section we compute the steady-state intensity dependent density and temperature profiles where collisional (inverse - bremsstrahlung) heating of the electrons is taken into account. As this process involves electron-ion collisions, the time scale required to establish an equilibrium is much longer than the acoustic transit time τ_{ac}, so we are dealing with a longer time scale process than in the case of the ponderomotive mechanism. We restrict the analysis to intensities such that the ponderomotive force discussed in the previous section may be neglected.

We begin by solving the Boltzmann equation;

$$\{2.20\} \quad \partial_t f + \vec{v}\cdot\partial_{\vec{x}} f + \vec{a}\cdot\partial_{\vec{v}} f + (\vec{\Omega}_e \times \vec{v})\cdot\partial_{\vec{v}} f = \delta f/\delta t$$

where $\vec{a}=-e\vec{E}/m_e$, $\vec{\Omega}_e=e\vec{B}/m_e$ and $\delta f/\delta t$ is a suitable collision term.

The solution of {2.20} will follow the formalism of Sharkrofsky et al (1966) whereby we expand the distribution function , f, in terms of spherical harmonics; $Y_{lm}(\Omega)$ ie.

$$f(\vec{v}) = \sum_{l,m,s} f_{lms}(v) Y_{lms}(\Omega)$$

$$= f_{000} + f_{100}\cos\theta + f_{110}\sin\theta\cos\varphi + f_{111}\sin\theta\sin\varphi + \ldots$$

$$= f_0 + \frac{\vec{f}_1 \cdot \vec{v}}{v} + \ldots \qquad \{2.21\}$$

where $f_0 = f_0(v)$ is the symmetric portion of the distribution function and

$$\vec{f}_1 = \begin{pmatrix} f_{110}(v) \\ f_{111}(v) \\ f_{100}(v) \end{pmatrix}$$

is the first order, asymmetric correction to the distribution function. Expansion {2.21} was used by Sodha <u>et al</u> (1974) in the investigation of self-focusing in a field-free ($B_0=0$) plasma.

According to Sharkrofsky <u>et al</u>, expansion {2.21} is useful if;

(i) in one wave period, the change in the electric field seen by the particle is small. This condition can be represented by

{2.22a} $$\left(\frac{v_0}{\omega L}\right)^2 << 1$$

where v_0 is the electron thermal velocity.

(ii) the electric field be such that the electron quiver velocity, $\tilde{v}=eE/m_e\omega$, is small compared to v_0, ie.

{2.22b} $$\left(\frac{\tilde{v}}{v_0}\right)^2 << 1$$

Substitution of the spherical harmonic expansion {2.21} into the Boltzmann equation {2.20} yields;

1. The l=0 scalar (density) equation

{2.23} $$\partial_t f_0 + \frac{1}{3}\partial_{\vec{x}}\cdot\vec{f}_1 + \frac{3}{v^2}(v^2\vec{a}\cdot\vec{f}_1) - \frac{m_e}{m_i \cdot v^2}\partial_v\left[v^3\nu\left(f_0 + \frac{T}{v m_e}\partial_v f_0\right)\right] = 0$$

where ν is the electron-ion collision frequency.

2. The l=1 vector (mometum) equation

$$\partial_t \vec{f}_1 + v \partial_{\vec{x}} f_0 + \vec{a}\, \partial_v f_0 + \vec{\Omega}_e \times \vec{f}_1 + \nu \vec{f}_1 = 0 \tag{2.24}$$

Due to the properties of the spherical harmonics, ensemble averages of scalar and vector quantities are defined as

$$\psi = \psi(v) \rightarrow \langle \psi \rangle = \frac{4\pi}{n} \int_0^\infty \psi f_0 v^2 dv$$

{2.25}

$$\vec{Q} = Q(v)\frac{\vec{v}}{v} \rightarrow \langle \vec{Q} \rangle = \frac{4\pi}{3n} \int_0^\infty Q(v) \vec{f}_1 v^2 dv$$

In the presence of time varying fields, we assume a time dependence of $e^{i\omega t}$ and expand the distribution function as a Fourier series;

$$f = \sum_k^\infty \left[f_0^k e^{ik\omega t} + \frac{\vec{f}_1^{\,k} \cdot \vec{v}\, e^{ik\omega t}}{v} + \dots + c.c. \right] \tag{2.26}$$

From {2.25}, we see that the density is given by

$$n = 4\pi \sum_{k} e^{ik\omega t} \int_0^\infty f_0^k v^3 dv \tag{2.27}$$

where the real part is to be taken. We can define $n^0, n^1, \ldots$ as

$$n^k e^{ik\omega t} = 4\pi e^{ik\omega t} \int_0^\infty f_0^k v^3 dv, \qquad k=0,1,2,\ldots \tag{2.28}$$

where the n^k, $k>1$ are associated with a.c. space charge effects. These effects can be neglected for the case where thermal velocities are much less than the phase velocity of the e.m. wave ($v_0 \ll \omega/k \sim c$) and we thereby set

$$n \equiv n^0 = 4\pi \int_0^\infty f_0^0 v^3 dv \tag{2.29}$$

and $n^k=0$ for $k>1$. From {2.25} we see that the average velocity is

$$\vec{v}^{\,k} = \frac{4\pi}{3n} e^{ik\omega t} \int_0^\infty \vec{f}_1^{\,k} v^3 dv \tag{2.30}$$

In the following treatment, we simplify the analysis by considering only the d.c. and fundamental harmonic terms (k=0,1) in the expansion for f. This is justified when the highest frequency of interest is the oscillation frequency of the e.m. wave since higher harmonic terms will have frequencies considerably removed from any other frequency of interest. This impies the frequency ordering $\omega^2 \gg \nu^2, \Omega_e^2, \omega_p^2$, etc.

We substitute the Fourier expansion {2.26} into the Boltzmann equation {2.20}, retaining only the k=0,1 terms and obtain the resulting density and momentum equations, which will each split into oscillatory and non-oscillatory parts. The l=0 (density) equation yields;

$$\frac{v}{3}\partial_x \cdot \vec{f}_1^{\,0} + \frac{1}{3v^2}\partial_v\left[v^2\left(\vec{a}^{\,0}\cdot\vec{f}_1^{\,0} + \frac{1}{2}\mathrm{Re}\left(\vec{a}^{\,1*}\cdot\vec{f}_1^{\,1}\right)\right)\right] -$$

$$- \frac{m_e}{m_i}\frac{1}{v^2}\partial_v\left[v^3\nu\left(f_0^{\,0} + \frac{T}{m_e v}\partial_v f_0^{\,0}\right)\right] = 0 \tag{2.31a}$$

$$i\omega f_0^1 + \frac{v}{3}\,\partial_{\vec{x}}\cdot\vec{f}_1^{\,1} + \frac{1}{3v^2}\,\partial_v\left[v^2\left(\vec{a}^{\,0}\cdot\vec{f}_1^{\,1} + \vec{a}^{\,1}\cdot\vec{f}_1^{\,0}\right)\right] -$$

{2.31b} $$- \frac{m_e}{m_i}\,\frac{1}{v^2}\,\partial_v\left[v^3\nu\left(f_0^1 + \frac{T}{m_e v}\,\partial_v f_0^1\right)\right] = 0$$

The momentum equation yields:

{2.32a} $$v\,\partial_{\vec{x}} f_0^0 + \vec{a}^{\,0}\partial_v f_0^0 + \frac{1}{2}\,\mathrm{Re}\left(\vec{a}^{\,1*}\partial_v f_0^1\right) + + \vec{\Omega}_e\times\vec{f}_1^{\,0} + \nu\vec{f}_1^{\,0} = 0$$

{2.32b} $$i\omega\vec{f}_1^{\,0} + v\,\partial_{\vec{x}} f_0^1 + \vec{a}^{\,1}\partial_v f_0^0 + \vec{a}^{\,0}\partial_v f_0^1 + + \vec{\Omega}_e\times\vec{f}_1^{\,1} + \nu\vec{f}_1^{\,1} = 0$$

In the above equations, $\vec{a}^{\,0}=-e\vec{E}^0/m_e$ where $\vec{E}^0$ is an electrostatic field (ie. due to charge separation) and $\vec{a}^{\,1}=-e\vec{E}^1/m_e$ where $\vec{E}^1$ is the amplitude of the e.m. wave. The neglect of a.c. space charge effects implies that $n^1=0$ which, from {2.28} implies that $f_0^1=0$. We therefore set $f_0^1=0$ in {2.31} and {2.32} and solve them for f_0^0, $\vec{f}_1^{\,0}$, and $\vec{f}_1^{\,1}$. Take the external magnetic field along the z-axis only, then from {2.32a} it can be shown that

{2.33} $$f^{\circ}_{ij} = -M_{ij}\left(v\,\partial_i f^{\circ}_{o} - \frac{e\,E^{\circ}_{i}}{m_e}\,\partial_v f^{\circ}_{o} \right)$$

where

{2.34} $$M^{\circ}_{ij} = \frac{1}{\nu^2+\Omega_e^2}\begin{bmatrix} \nu & -\Omega_e & 0 \\ \Omega_e & \nu & 0 \\ 0 & 0 & \dfrac{\nu^2+\Omega_e^2}{\nu} \end{bmatrix}$$

From {2.32b} it can be shown that

{2.35} $$f^{1}_{ij} = M^{1}_{ij}\; e\,\frac{E^{1}_{i}}{m_e}\,\partial_v f^{\circ}_{o}$$

where

$$\{2.36\} \qquad M^{1}_{ij} = \frac{1}{(\nu+i\omega)^2+\Omega_e^2}\begin{bmatrix} \nu+i\omega & -\Omega_e & 0 \\ \Omega_e & \nu+i\omega & 0 \\ 0 & 0 & \dfrac{(\nu+i\omega)^2+\Omega_e^2}{\nu+i\omega} \end{bmatrix}$$

Substitution of {2.33} and {2.35} into {2.32a} will yield an equation for the symmetric part of the distribution function $f_0^0(v)$. We write the gradient operator as $\vec{\nabla} = \partial_{\parallel} + \partial_{\perp}$ where the directions are with respect to the external magnetic field $\vec{B}_0$. If the following assumptions are made,

1. linearly polarized e.m. wave
2. $E^0 \ll E^1$ (or scale length >> Debye length)
3. $\omega^2 \gg \Omega_e^2 \gg \nu^2$

the equation for f_0^0 becomes

{2.37}

$$\frac{v}{3}\left\{\frac{\nu}{\nu^2+\Omega_e^2}\left[v\nabla_\perp^2 f_0^\circ+\frac{\partial_\perp\Omega_e^2}{\nu^2+\Omega_e^2}\cdot v\,\partial_{\bar{x}} f_0^\circ\right]+\frac{v}{\nu}\nabla_\parallel^2 f_0^\circ\right\}+$$

$$+\frac{\vec{\Omega}_e}{\nu^2+\Omega_e^2}\cdot\left(\frac{\nu^2}{\Omega_e^2}-1\right)\left[\frac{\partial_\perp\Omega_e^2}{\nu^2+\Omega_e^2}\times\partial_{\bar{x}}\left(\partial_v f_0^\circ\right)\right]+$$

$$+\frac{1}{3v^2}\partial_v\left\{v^3\nu\left(w\,\partial_u f_0^\circ+\frac{3}{2}f_0^\circ\right)\right\}=0$$

where $$W=\frac{3}{2}T_0+\frac{m_i}{\omega^2}\frac{(a')^2}{4}=\frac{3}{2}T_0\left(1+\frac{m_i\cdot e^2E'^2}{6m_e\omega^2T_0}\right)$$

$$u=\frac{1}{2}m_e v^2$$

In order to simplify the above equation, we dimensionally investigate the ordering of terms in {2.37}. Taking the scale length for perpendicular gradients to be L, we find that

term (1): $$\frac{v^2\nu f_0^\circ}{3\Omega_e^2L^2}\sim\left(\frac{a_e}{L}\right)^2\frac{\nu}{3}f_0^\circ$$

term (2): $$\frac{v^2\nu f_0^\circ}{3\Omega_e^2L^2}\sim\left(\frac{a_e}{L}\right)^2\frac{\nu}{3}f_0^\circ$$

term (3): $\nabla_\parallel^2 f_0^\circ=0$ (assuming f_0° uniform along $\vec{B}_0$)

term (4): by symmetry the cross product vanishes

term (5): $\sim \dfrac{\nu \xi f_0^{\circ}}{3}$

where $\xi = \frac{2m_e}{m_i + m_e}$ and a_e is the electron Larmor radius. Thus the ordering of the surviving terms is

{2.38} $$\left(\frac{a_e}{L}\right)^2 \; : \; \left(\frac{a_e}{L}\right)^2 \; : \; \xi$$

Therefore the neglect of terms (1) and (2) in {2.37} requires

$$\left(\frac{a_e}{L}\right)^2 << \xi$$

Since $\xi \sim 2m_e/m_i$, we can write this condition as

{2.39} $$\left(\frac{a_i}{L}\right)^2 << 1$$

Thus the ion Larmor radius must remain much smaller than any scale lengths of interest, and this implies that radial thermal conduction losses from regions of size $\sim$ L will be negligible.

Assuming that {2.39} holds, the equation for f_0° is

{2.40} $$\frac{1}{3} v^2 \nu \left(\frac{2}{3} w \partial_u f_0^\circ + \frac{3}{2} f_0^\circ \right) = 0$$

The first integral of {2.40} is

{2.41} $$u^{3/2} \nu \left(\frac{2}{3} w \partial_u f_0^\circ + f_0^\circ \right) = C$$

Since f_0° and $\partial_u f_0^\circ \longrightarrow 0$ as $u \longrightarrow \infty$ (more rapidly than $u^{3/2} \longrightarrow \infty$) then as {2.41} must hold for all u, C must be zero to satisfy the limit $u \rightarrow \infty$. Thus we must solve

$$T_0 \left(1 + \frac{m_i e^2 E^2}{6 m_e \omega^2 T_0} \right) \partial_u f_0^\circ + f_0^\circ = 0$$

The solution for f_0° is

{2.42} $$f_0^\circ = A \exp\left[- \frac{m_e v^2}{2 T_E} \right]$$

where $T_E = T_0 \left(1 + \alpha_h E^2 \right)$, $\alpha_h = m_i e^2 / 6 m_e^2 \omega^2 T_0$

From the normalization condition $n_e = 4\pi \int_0^\infty v^2 f_0^\circ \, dv$, we find

that

$$A = n_e \left(\frac{m_e}{2\pi T_E} \right)^{3/2}$$

so that

$$f_0^{\circ} = n_e \left(\frac{m_e}{2\pi T_E} \right)^{3/2} \exp\left(-\frac{m_e v^2}{2 T_E} \right) \tag{2.43}$$

The above distribution function is Maxwellian with an effective electron temperature $T_E = T_0(1+\alpha_h E^2)$. Using {2.43}, f_0^1 and $\vec{f}_1^{\,1}$ may be computed.

We now derive an equation for the electron density from the above results considering only the steady-state, which is defined to be when the zero-order (d.c.) current, $\vec{J}^0$, is zero (ie. $\vec{v}^0=0$, see {2.30}). Setting

$$\vec{J}^{\circ} = \frac{4\pi}{3} e \int_0^{\infty} v^3 \vec{f}_1^{\,\circ} \, dv = 0 \tag{2.44}$$

and using {2.43} and {2.33} in {2.44} yields (see Appendix A)

$$\partial_r (n_e T_E) = -2 n_e e E^{\circ} \tag{2.45}$$

This is the equilibrium equation for electrons.

We can repeat the entire procedure for ions and solve for f°_{oi} etc. and obtain exactly the same equation as {2.45} except that T_E would be replaced by $T_i = T_o$ (a constant), because of the assumption that the ions are not heated by the e.m. field. Thus for ions the equilibrium equation would be ($e \rightarrow -e$)

$$T_o \partial_r n_i = 2 n_i e E^{\circ} \tag{2.46}$$

We solve for n_e by assuming that scale lengths are much larger than a Debye length so that we have the quasi-neutrality condition $n_e \approx n_i$, and eliminate E° from {2.45} and {2.46} to obtain

$$n_e = C_o / T_E + T_o \tag{2.47}$$

Now $n_e \rightarrow n_o$ as $T_E \rightarrow T_o$ (ie. as $E^2 \rightarrow 0$) so that $C_o = 2T_o n_e$

and

{2.48} $$\eta = \frac{n_e}{n_o} = \left(1 + \alpha_h E^2/2\right)^{-1}$$

is the equilibrium density profile in the presence of the em field in a magnetized plasma. In the unmagnetized case, collisions play a dominant role and Sodha et al (1974) have found the equilibrium profile to be, by a similar kinetic theory procedure,

{2.49} $$\eta = \left(1 + \alpha_h E^2/2\right)^{-5/2}$$

Thus there is a significant increase in the depth of the density well in the $B_o=0$ case as expected. It must be stressed however, that {2.48} is not valid for arbitrary magnetic fields (B_o) and intensities (E^2). The assumptions used to derive {2.48} restrict the values of these parameters and we review these assumptions below.

The conditions for validity of {2.48}, along with assumptions about symmetry, are

1. $\xi_1 = (v_o \tau/L)^2 \ll 1$

2. $\xi_2 = (\tilde{v}/v_o)^2 \ll 1$

3. $\nu^2 \ll \Omega_c^2 \ll \omega^2$

4. $\xi_4 = (a_i/L)^2 << 1$

Conditions 1 and 2 are required for the validity of the truncation of the expansion of the distribution function whereas the last two conditions simplified the analysis. Conditions 2 and 4 are the most stringent and table 1 shows the values of parameters required for validity for various initial temperatures. We assume CO_2 radiation, and the scale length L to be the beam size, L~0.1 cm typically.

The results obtained here (density and temperature profiles) have been compared with those from a one-dimensional MHD code (Milroy (1976)) and found to agree well (see Figure 1).

We now compute the non-linear dielectric explicitly using the formalism of the expansion scheme of Sharkrofsky <u>et al</u> (1966). From the definition of the a.c. conductivity σ_{ij}^1, we have that

{2.50} $$J_j^1 = \sigma_{ij}^1 E_i^1 = \frac{4\pi}{3} e \int_0^\infty v^3 \vec{f}_1^1 \, dv$$

Substituting for $\vec{f}_1^1$ from {2.35} we see that σ_{ij}^1 can be written as

{2.51} $$\sigma_{ij}' = \frac{4\pi}{3}\frac{e}{m_e}\int_0^\infty M_{ij}^1 \, v^3 \partial_v f_0^0 \, dv$$

Using the ordering $\nu^2 \ll \Omega_e^2 \ll \omega^2$, the components of M_{ij}^1 are

$$M_{11}' = M_{22}' = M_{33}' \simeq \frac{\nu - i\omega}{\omega^2}$$

$$M_{12}' = -M_{21}' \simeq \frac{\Omega_e}{\omega}\left(\frac{\omega + i\,2\nu}{\omega^2}\right)$$

$$M_{13}' = M_{31}' = M_{23}' = M_{32}' = 0$$

Putting the above expressions to use in {2.51} together with the expression derived earlier for f_o^o gives, after integration

{2.52} $$\sigma_{11}' = \sigma_{22}' = \sigma_{33}' = -\frac{i e^2 n_o}{m_e \omega} + \frac{4e^2\nu}{3\sqrt{\pi}\, m_e \omega^2}\left(1+\alpha_h E^2\right)^{-3/2}$$

$$\{2.53\} \quad \sigma_{12}' = -\sigma_{21}' = \frac{i\, 8 e \nu n_e \Omega_e}{3\sqrt{\pi}\, m_e \omega^2 (1+\alpha_h E^2)^{3/2}} + \frac{e n_e \Omega_e}{m_e \omega^2}$$

and all other components vanish. Now the components of the dielectric tensor are

$$\{2.54\} \quad \epsilon_{ij} = \delta_{ij} - i\frac{4\pi}{\omega}\sigma_{ij}'$$

and this becomes, using {2.52} and {2.53},

$$\{2.54\} \quad \epsilon_{ij} = \begin{bmatrix} \epsilon_1 & \epsilon_2 & 0 \\ -\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_1 \end{bmatrix}$$

where $\epsilon_1 \equiv \epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \epsilon_L + \Phi(E) + i\epsilon'$

$$\{2.56\} \quad \epsilon_2 = \frac{2\Omega_e}{\omega}\epsilon' + i\frac{\Omega_e}{\omega}\left[\Phi - \omega_p^2/\omega^2\right]$$

$$\{2.57\} \quad \epsilon_L = 1 - \omega_p^2/\omega^2$$

$$\Phi(E) = \frac{\omega_p^2}{\omega^2}(1-\eta)$$

$$\{2.58\} \quad \epsilon' = -\frac{4}{3\sqrt{\pi}}\frac{\nu}{\omega}\frac{\omega_p^2}{\omega^2}\eta\left(1+\alpha_h E^2\right)^{-3/2}$$

This non-linear dielectric will be used later in the solution of the wave equation.

2.3 COLLISIONAL HEATING - FLUID THEORY

In this section we try an approach that is less restricted than the more rigorous kinetic approach used in the previous section. For example, the the kinetic approach did not allow for ion heating or for radial thermal conduction whereas the following theory removes these restrictions.

We start from the static fluid equations where inertial terms are neglected (ie. $\partial_t = 0$) which is a valid assumption if we are solving for the equilibrium state (ie. for long time scales). The momentum equation becomes a pressure balance equation (B_0 is the external magnetic field),

$$n\left(T_e + T_i\right) + \frac{B^2}{8\pi} = \frac{B_0^2}{8\pi} \tag{2.59}$$

The electrical conductivity of the plasma is assumed to be infinite, valid at high temperatures, which gives the frozen field condition

{2.60} $$\eta = \frac{n}{n_o} = \frac{B}{B_o}$$

The electron and ion energy equations are;

{2.61} $$\frac{2}{3}\frac{I_L k_\nu}{n} + \frac{2}{3}\frac{1}{n}\frac{1}{r}\partial_r\left(\chi_e r \partial_r T_e\right) - \frac{T_e - T_i}{\tau_{eq}} = 0$$

{2.62} $$\frac{2}{3}\frac{1}{n}\frac{1}{r}\partial_r\left(\chi_i r \partial_r T_i\right) + \frac{T_e - T_i}{\tau_{eq}} = 0$$

In the above equations, I_L is the laser intensity (W/cm²), k_ν is the inverse-bremsstrahlung absorption coefficient, $\chi_{e,i}$ are the electron, ion thermal conductivities and τ_{eq} is the e-i equilibration time.

We accomodate the ion heating by assuming a fixed ion-electron temperature ratio for all spatial points, ie.

{2.63} $$T_i = \theta T_e \quad , \quad 0 \le \theta \le 1$$

This assumption is related to the temporal structure of the laser profile (Vagner <u>et al</u> (1975)).

Using {2.60} and {2.63} in the pressure balance equation, we obtain an expression for the density,

{2.64} $$\eta = \left[(1+\theta)^2 \beta_o^2/4 + 1\right]^{1/2} - (1+\theta)\beta_o/2$$

where we have defined

{2.65} $$\beta_o = \frac{8\pi n_o T_e}{B_o^2}$$

For strong B-fields, $\beta_o \ll 1$ and {2.64} becomes

{2.66} $$\eta \simeq 1 - (1+\theta)\beta_0/2$$

We now have η as a function of electron temperature, so we must use the energy conservation equations to solve for T_e in terms of the input laser power. Adding {2.61} to {2.62}, using {2.60}, we obtain an equation governing T_e,

{2.67} $$\partial_{rr} T_e + r^{-1}\partial_r T_e + \partial_r T_e \frac{\partial_r \chi_i}{\chi_i} + \frac{I_L k_\nu}{\theta \chi_i} = 0$$

where we have assumed that the electron radial thermal conductivity is much smaller than the ion thermal conductivity in magnetic fields of interest. We can rewrite the $\partial_r \chi_i$ term in {2.67} by noting that $\chi_i = C/T_i^{1/2}$, therefore $\partial_r \chi_i = -\frac{C\theta\partial_r T_e}{2(\theta T_e)^{3/2}} = -\frac{1}{2}\frac{\chi_i \partial_r T_e}{T_e}$ and {2.67} becomes

{2.68} $$\partial_{rr} T_e + r^{-1}\partial_r T_e - \frac{1}{2} T_e^{-1}(\partial_r T_e)^2 + (\theta\chi_i)^{-1} I_L k_\nu = 0$$

The laser intensity is taken to be Gaussian with beam width σ_0 so that we have

{2.69} $$I_L = I_0 \exp(-r^2/2\sigma_0^2)$$

where I_0 (w/cm²) is the on-axis laser intensity. The inverse-bremsstrahlung absorption coefficient and ion thermal conductivity (Milroy (1976) and Spitzer (1967) respectively) are

$$k_{\nu} = \frac{8.67 \times 10^{-31}\, n^2 Z \lambda^2(\mu) \ln \Lambda}{T_e^{3/2}} \tag{2.70}$$

$$\chi_i = 1.6 \times 10^{-15} \frac{n_o^2}{B_o^2} (\theta T_e)^{-1/2} \ln \Lambda \tag{2.71}$$

where λ is the laser wavelength in microns and $\ln\Lambda$ is the Coulomb logarithm.

Using {2.69} to {2.71} the last term in {2.68} can be written as

$$\frac{I_L k_\nu}{\theta \chi_i} = \frac{A(r)}{T_e} \exp(-r^2/2\sigma_o^2) \tag{2.72}$$

where Z=1, and

$$A(r) = 5.4 \times 10^{-16} \lambda^2(\mu)\, B_o^2\, \eta^2(r)\, \theta^{-1/2} I_o \tag{2.73}$$

We solve {2.68} by assuming the form of the electron temperature profile to be Gaussian, i.e. ,

$$T_e = T_1 \exp(-r^2/2\sigma_1^2) + T_o \tag{2.74}$$

This assumption is reasonable since the laser intensity is Gaussian and a form similar to {2.74} was obtained in section 2.2 using kinetic theory. Equation {2.74} has three parameters to solve for; the width σ_1, the maximum on-axis temperature T_1, and the temperature far away from the axis

T_0. To simplify the analysis, we restrict ourselves to the region near the axis and assume $T_1 \gg T_0$ so that we may use the approximate form of the temperature profile

$$T_e = T_1 \exp\left(-r^2/2\sigma_1^2\right) \tag{2.75}$$

Substituting {2.75} into {2.68} yields the equation

$$\frac{r^2}{2\sigma_1^2} + \frac{\sigma_1^2 A(r)}{T_e^2}\exp\left(-r^2/2\sigma_0^2\right) - 2 = 0 \tag{2.76}$$

We set r=0 in {2.76} and obtain an expression for the on-axis temperature T_1,

$$T_1^2 = \frac{\sigma_1^2}{2} A(0) \tag{2.77}$$

Differentiating {2.76} twice with respect to r and then setting r=0 yields a relation for the temperature profile width σ_1 (see Appendix B)

$$\sigma_1^2 \simeq \frac{5}{2}\sigma_0^2 \tag{2.78}$$

Thus the temperature profile is wider than the laser intensity profile by $\sqrt{5/2}$, which is a more realistic situation than that found in section 2.2 where the widths were the same. Equation {2.78} shows the effect of a non-

zero radial thermal conductivity. Note that {2.77} and {2.78} are only valid for $T_1 >> T_0$.

Using {2.77} and {2.78} in {2.64} we can write the density profile as

{2.79} $$\eta=\left[1+\alpha_s^2\eta^2(0)E_0^2\exp(-r^2/2\sigma_1^2)\right]^{1/2}-\alpha_s\eta(0)E_0\exp(-r^2/2\sigma_1^2)$$

where $\alpha_s = 5.23\times10^{-19}\sigma_0\dfrac{n_0}{B_0}\lambda(1+\theta)\theta^{-1/4}$

Setting r=0 in {2.79}, we obtain an expression for the on-axis density ratio $\eta(0)$,

{2.80} $$\eta(0)=(1+2\alpha_s E_0)^{-1/2}$$

Equation {2.79} can now be written as

{2.81} $$\eta(r)=\left[1+X^2(r)\right]^{1/2}-X(r)$$

where

{2.82} $$X(r)=\frac{\alpha_s E_0}{(1+2\alpha_s E_0)^{1/2}}\exp(-r^2/2\sigma_1^2)$$

The real part of the dielectric can now be written as,

for $\omega^2 \gg \Omega_e^2$,

$$\epsilon = \epsilon_L + \Phi(x) \tag{2.83}$$

where

$$\Phi(x) = \frac{\omega_{p-}^2}{\omega^2}\left(1 - \eta(x)\right) \tag{2.84}$$

The on-axis (electron) temperature T_1 and density ratio $\eta(0)$ are plotted in Figure 2 for various laser powers.

3. SOLUTION OF THE WAVE EQUATION WITH A NONLINEAR DIELECTRIC CONSTANT

In this chapter we solve the wave equation using a WKB - solution which was developed by Akmanov et al (1966) for the study of e.m. wave propagation in liquids and crystals with nonlinear dielectrics. As was mentioned in the introduction, we assume the frequency ordering $\omega^2 >> \Omega_e^2$, such that we can neglect the non-vanishing off-diagonal elements of the dielectric tensor because these terms are of $O(\Omega_e^2/\omega^2)$ and hence very small. We thereby study linearly polarized solutions of the wave equation with a dielectric of the form

{3.1} $$\epsilon = \epsilon_L + \Phi + i\epsilon'$$

where

{3.2} $$\Phi = \frac{\omega_{p}^2}{\omega^2}(1-\eta)$$

and the effect of the magnetic field is to alter the density profile. The imaginary part of the dielectric is assumed to be much smaller than the real part since $\omega^2 >> \nu^2$.

The wave equation is

$$\{3.3\} \qquad \nabla^2 \vec{\mathcal{E}} + \vec{\nabla}\left[\vec{\mathcal{E}} \cdot \frac{\vec{\nabla}\epsilon}{\epsilon}\right] + \frac{\omega^2}{c^2}\epsilon \vec{\mathcal{E}} = 0$$

Equation {3.3} can be analyzed by employing the concept of slowly varying amplitudes (Akmanov <u>et</u> <u>al</u> (1966)) whereby a weakly converging (or diverging) beam can be represented by

$$\{3.4\} \qquad \vec{\mathcal{E}} = \hat{e}\,\frac{1}{2}\left[\mathcal{E}(r,z,s)\exp\left[-i(\omega t - kz)\right] + c.c.\right]$$

where $\hat{e}$ is a unit polarization vector and s is a parameter which accounts for the difference between the actual beam and a plane wave state. The difference is due of course to the action of the nonlinearity or to diffraction. Substitution of {3.4} into {3.3} yields,

$$\{3.5\} \qquad \left(\nabla_\perp^2 + \partial_{zz} + 2ik\partial_z - k^2 + \frac{\omega^2}{c^2}\epsilon\right)\mathcal{E}(r,z,s) = 0$$

where we have neglected $\vec{\nabla}[\vec{\mathcal{E}} \cdot \frac{\vec{\nabla}\epsilon}{\epsilon}]$ compared to $k^2\mathcal{E}$ which is valid for $\omega^2 \gg \omega_p{}^2$. We solve {2.5} with cylindrical symmetry ($\nabla_\perp{}^2 = \partial_{rr} + r^{-1}\partial_r$) and by neglecting $\partial_{zz}\mathcal{E}$ compared to

$k^2\mathcal{E}$. This latter assumption implies that the scale length of variation of the amplitude of the electric field along the direction of propagation must be greater than the wavelength of the radiation field (i.e. slowly varying amplitude assumption). Following Akmanov <u>et al</u> we take the WKB-solution for the amplitude to be

$$\mathcal{E} = E(r,z)\, e^{-ik\,s(r,z)} \tag{3.6}$$

where s=s(r,z) is the addition to the eikonal which accounts for the nonlinearity in the dielectric and E(r,z) is the amplitude of the e.m. wave. Substitution of {3.6} into {3.5}, with $\partial_{zz}\mathcal{E} \ll k^2\mathcal{E}$ yields an equation that can be split into real and imaginary parts as

$$\partial_{rr}E - k^2(\partial_r s)^2 E + r^{-1}\partial_r E - 2k^2 E\,\partial_z s - k^2 E + \omega^2 c^{-2}(\epsilon_L + \Phi)E = 0 \tag{3.7a}$$

$$kE^2\partial_{rr}s + k\,\partial_r E^2\,\partial_r s + r^{-1}kE^2\partial_r s + k\,\partial_z E^2 + \omega^2 c^{-2}\epsilon' E^2 = 0 \tag{3.7b}$$

Equations {3.7a} and {3.7b} are similar to those obtained in the study of lens-like (linear) media whose solutions are cylindrical or spherical converging (or diverging) waves. The difficulty in the above equations is that the amplitude E(r,z) and the eikonal s(r,z) are coupled by the nonlinearity. Following Akmanov et al, the solution of {3.7} for s(r,z) will be generalized to cylindrical or spherical waves with variable radius of curvature whereby we take

{3.8} $$s(r,z) = \frac{r^2}{2}\beta(z) + \varphi(z)$$

where $\beta(z)$ is interpereted as the radius of curvature, and $\varphi(z)$ is the change in the phase of the eikonal due to the nonlinear medium. The solution for the amplitude will be obtained by employing a Gaussian- shaped ansatz,

{3.9} $$E^2 = \frac{E_o^2}{f^2(z)} \exp\left(-r^2/a^2 f^2(z)\right) F^2(z)$$

where a is the 1/e beam size at z=0, f(z) is a dimensionless beam-size parameter, F(z) accounts for the absorption by the medium, and E_o^2 is the on-axis field intensity. Equation {3.9} assumes that the laser intensity profile remains Gaussian at all times, however the on-axis intensity $E_o^2/f^2(z)$ and beam width af(z), change as the beam propagates through the medium.

In the following dicussions we will neglect the effect

of absorption over length scales of interest. This will be valid if the plasma is sufficiently hot so that classical inverse-bremsstrahlung absorption is weak, i.e. the plasma is optically thin over length scales of interest. The neglect of absorption implies that the imaginary part of the dielectric (i.e. ϵ' in {3.1}) can be neglected compared to the real part. The discussion is exact for the case of the ponderomotive nonlinearity if no collisionless absorption process occurs.

We substitute {3.8} and {3.9} into {3.7a} and {3.7b} with F(z)=1 (i.e. no absorption) and obtain from {3.7b},

$$\beta(z) = f^{-1} \partial_z f(z) \tag{3.10}$$

and from {3.7a}, using {3.10},

$$\left\{ \frac{r^2}{a^4 f^4} - \frac{2}{a^2 f^2} - 2k^2 \left(\frac{r^2}{2} f^{-1} \partial_{zz} f + \partial_z \varphi \right) - k^2 + \right. \\ \left. + \omega^2 c^{-2} \left(\epsilon_L + \Phi \right) \right\} E = 0 \tag{3.11}$$

As {3.11} must hold for all r and z, in particular r=0, we set r=0 in {3.11} to get an equation for the phase $\varphi(z)$,

$$\partial_z \varphi(z) = -\frac{1}{k^2 a^2 f^2} + \frac{\omega^2 \Phi - \Gamma^2}{2c^2 k^2} \tag{3.12}$$

where

{3.13} $$\Gamma^2 = c^2k^2 - \omega^2 + \omega_p^2$$

The quantity Γ represents the nonlinear wave number shift due to the change in the on axis density.

In order to obtain an equation for the beam width parameter f(z), we differentiate {3.11} twice with respect to r and set r=0, and use {3.12} to get, after some manipulation,

{3.14} $$\partial_{zz} f(z) = \begin{cases} \dfrac{1}{k^2a^2f^3}\left[\dfrac{1}{a^2} - \dfrac{\omega^2 E_0^2}{c^2}\Phi'\left(\dfrac{E_0^2}{f^2}\right)\right] \\ \\ \dfrac{1}{k^2a^2f^3}\left[\dfrac{1}{a^2} - \dfrac{\omega^2 f^2 X(0)}{c^2}\Phi'(X(0))\right] \end{cases}$$ {3.15}

where {3.14} is for the ponderomotive and weak heating cases discussed in 2.1 and 2.2 where Φ was a function of E^2. Equation {3.15} applies to the strong heating case discussed in 2.3 where $\Phi = \Phi(X)$. In the above equations, the prime denotes differentiation of Φ with respect to its arguement.

Equation {3.14} and {3.15} describe the propagation of the laser beam through the plasma over distances where absorption is not important. We will refer to the differential equation for f(z) as the self-focusing equation

for the particular mechanism involved (i.e. ponderomotive, weak or strong heating). The beam will converge (i.e. the effective beam-width $af(z)$ decreases) only if the slope of $f(z)$ is a decreasing function of z, i.e. $\partial_z f(z)<0$. By inspection of the self-focusing equations, we see that this condition is only met if the second term on the right-hand-side is larger than the first. The second term is due to the nonlinearity in the dielectric and is thus the focusing term while the first term is the diffraction term and if it dominates the focusing term , then the beam will tend to diverge (i.e. $\partial_z f>0$). A steady state can occur if the two terms balance exactly.

4. SELF-FOCUSING IN A MAGNETOPLASMA

In this chapter, we investigate solutions to the equations derived in Chapter 3 which described self-focusing i.e. equations {3.12}, {3.14} and {3.15}. These equations do not admit an easy analytic solution except under special conditions, however a good deal of information can be obtained without resorting to a numerical solution.

4.1 STEADY-STATE SELF-FOCUSING - SELF-TRAPPED SOLUTIONS

In this section we investigate self-trapped (or soliton like) solutions where the beam propagates in a steady-state manner (i.e. with no convergence or divergence). This occurs when the self-focusing process is exactly balanced by diffraction. The ponderomotive and weak heating cases will be considered first where the nonlinear part of the dielectric Φ, is a function of E^2, followed by a treatment of the strong heating case .

In the steady-state case we take $a \longrightarrow a_E$ (a_E will denote the equilibrium beam size), $f(z)=1$ and all derivatives vanish ($\partial_z f = \partial_{zz} f = \partial_z \varphi = 0$). From {3.12} we obtain for the wavenumber shift

$$\Gamma_E^2 = \omega^2 \Phi - 2c^2/a_E^2 \qquad \{4.1\}$$

and from {3.14} we obtain for the beam size

{4.2} $$a_E^2 = \frac{c^2}{\omega^2 E_0^2} \frac{1}{\Phi'(E_0^2)}$$

Equation {4.1} is the dispersion relation for the nonlinear medium for equilibrium propagation. This can be seen more clearly if we substitute the expressions for Γ^2 and a_E into {4.1},

{4.3} $$c^2 k^2 = \omega^2 - \omega_{p_0}^2 \eta(0) - 2c^2 a_E^{-2}$$

where $\eta(0)$ is the on-axis density ratio. This dispersion relation resembles that obtained for plasma filled waveguides where the last term is the correction due to the fact that the beam is bounded (i.e. self-trapped), and the second term shows the effect of the plasma depletion in the presence of the beam. Thus the beam appears to propagate in a self made waveguide, or light pipe.

Equation {4.2} gives the beam size, (as a function of the on-axis intensity E^2) that is required for self-trapped propagation. With $a=a_E$, the beam propagates uniformly through the plasma with no change in size. For a given intensity, this gives the critical power required for self-trapped propagation ($P_{cr} \propto a_E^2 E_0^2$)

{4.4} $$P_{cr} = 3.0 \times 10^3 a_E^2 E_0^2 \qquad (\text{Watts})$$

From {4.2} we can obtain a condition for the equilibrium beam size, a_E, to reach a minimum by setting $da_E^2/dE_o^2=0$. This gives

$$E_o^{-2} = -\Phi''/\Phi' \tag{4.5}$$

as the condition for the minimum equilibrium beam size a to occur. It will be shown later that the existence of $a_E)_{min}$ is a result of the limitation on the beam size due to diffraction effects and saturation of the nonlinearity.

We now investigate these equilibrium quantities for the cases discussed in Chapter 2.

a) Ponderomotive nonlinearity in a strong magnetic field.

We treat the case where β_o<<1 so that some analytical progress can be made. From {2.19} we have that

$$\Phi(E_o^2) = \frac{\omega_{p-}^2 \beta_o \alpha_r E_o^2}{2\omega^2}, \qquad \beta_o << 1 \tag{4.6}$$

so that Φ is parabolic in E . From {4.2} we see that for this case

$$a_E^2 = \frac{2c^2}{\omega_{p-}^2 \beta_o \alpha_r E_o^2} \tag{4.7}$$

The equilibrium beam size for the field free ($B_0=0$) case has recently been obtained (Max (1976)) and was found to be,

$$a_E^2 = \frac{2c^2 \exp[\alpha_r E_0^2/2]}{\omega_{p}^2 \alpha_r E_0^2} \tag{4.8}$$

Thus the equilibrium beam size is much larger for the magnetized plasma than for an unmagnetized plasma when $\alpha_r E_0^2 \simeq 1$. The magnetic field case requires more power to achieve self-trapped propagation than the field free case because the magnetic field reduces the plasma depletion in the beam and hence reduces the 'strength' of the self-focusing effect. A larger beam size is therefore required to reduce the diffraction effect (which is proportional to $1/a^2$) to balance the weaker self-focusing.

The critical power is found from {4.9} to be

$$P_{cr} = 6.0 \times 10^7 \frac{c^2}{\omega_{p} \beta_0 \alpha_r} \tag{4.9}$$

which gives, for CO_2 laser, $\beta_0 \sim 0.1$, $\omega_{p}^2/\omega^2 \sim 0.1$ and $T_e \sim$ 100eV a $P_{cr} \simeq 2 \times 10^{11}$ Watts.

b) Weak heating in a magnetic field.

From {2.48} we have that

$$\Phi = \frac{\omega_{p}^2}{\omega^2}\left[1 - \left(1 + \frac{\alpha_h E_0^2}{2}\right)^{-1}\right] \tag{4.10}$$

which for $\alpha_h E_0^2 << 1$, gives the parabolic form

$$\Phi \simeq \frac{\omega_p^2}{\omega^2} \frac{\alpha_h E_0^2}{2} \tag{4.11}$$

The equilibrium beam size is

$$a_E^2 = 2 \frac{c^2}{\omega_p^2} \frac{\left(1 + \alpha_h E_0^2/2\right)^{+2}}{\alpha_h E_0^2/2} \tag{4.12}$$

which for $\alpha_h E_0^2 << 1$, gives the critical power

$$P_{cr} = 6.0 \times 10^3 \frac{c^2}{\omega_p^2 \alpha_h} \tag{4.13}$$

For CO_2 laser, $T_e \sim 100$eV, $\omega_p^2/\omega^2 \sim 0.1$, this gives a $P_{cr} \sim 10^7$ W. Thus the ponderomotive force in a strong magnetic field requires considerably more power than the heating nonlinearity in order to achieve the self-trapped state. This is due to the ponderomotive force being inherently weak until high intensities are achieved. We thus conclude that the heating of the plasma and subsequent density depletion is by far the more dominant mechanism for self-focusing for laser intensities of interest in present day laser - plasma experiments.

As the heating nonlinearity is not parabolic for $\alpha_h E_0^2 \sim 1$, we can compute $a_E)_{min}$ from {4.10} using {4.5}. The condition for minimum equilibrium beam size to occur is

$$\alpha_h E_0^2 = 2 \tag{4.14}$$

Substitution of {4.14} into {4.12} yields

{4.15} $$a_E\big)_{min} = 2\, c/\omega_{p}$$

which is of the order of a few vacuum wavelengths across (for $\omega_p^2 \lesssim \omega^2$). Similar results have been found for the ponderomotive and weak heating cases in the field-free (B_0=0) situation. Max (1976) and Sodha et al (1974) find $a_E\big)_{min} \sim c/\omega_p$ as the typical scale length for these situations even though the mechanisms for the nonlinearity are different. Also, Kaw et al (1973) find that scale lengths of the order of c/ω_p across the beam are the smallest scale lengths for which the filimentation instability will occur. Max (1976) has suggested that the self-trapped states, which are equilibrium states, represent the final stage of the filimentation instability because the scale lengths are the same. We discuss the validity of obtaining such small scale lengths across the beam in a later chapter.

It is interesting at this point to examine the dispersion relation {4.3} for a cut-off beam size, $a_E\big)_{c/o}$ whereby c^2k^2=0, and solutions to the wave equation do not propagate. This cut-off is

{4.16} $$a^2_E\big)_{c/o} = \frac{2c^2/\omega^2}{1 - \omega_p^2/\omega^2\ \eta(0)}$$

Thus $a_E\big)_{c/o} \sim \lambda$ for $\omega_p^2/\omega^2 \ll 1$ so that a cut-off can only occur for beam sizes of the order of a vacuum wavelength.

As we have seen that $a_E)_{min} \sim c/\omega_{p-}$, then $a_E)_{cf} < a_E)_{min}$ and cut-offs will therefore not exist for $\omega_{p-}^2/\omega^2 \ll 1$.

c) Strong heating in a magnetic field.

At this point we solve for the equilibrium solutions to equations {3.12} and {3.15} for the strong heating case studied in 2.3. Setting $f(z)=1$ and $\partial_z f=0$ in {3.15}, we obtain for the equilibrium radius

$$a_E^2 = \frac{c^2}{\omega^2} \frac{1}{\chi(0)\, \Phi'(\chi(0))}$$

$$= \frac{c^2}{\omega^2} \frac{(1+\alpha_s E_0)(1+2\alpha_s E_0)^{1/2}}{\alpha_s E_0} \qquad \{4.17\}$$

The dispersion relation obtained from {3.12} by setting $\partial_z\varphi=0$ is identical to {4.3} except that Φ is given by {2.84}. From {4.17}, we find that the minimum equilibrium beam size occurs for

$$\alpha_s E_0 = \frac{1+\sqrt{5}}{2} \qquad \{4.18\}$$

and this gives

$$a_E)_{min} = 2.9\, c/\omega_{p-} \qquad \{4.19\}$$

So again the typical minimum beam size is of the order of

c/ω_p as found in the previous discussion on weak heating in part (b).

4.2 NONEQUILIBRIUM SELF-FOCUSING - WEAK FOCUSING LIMIT

We now examine the physical significance of the equilibrium beam size by discussing the cases for which the function Φ is parabolic in E_o, for example the ponderomotive case for $\beta_o \ll 1$ or the weak heating case when $\alpha_h E_o^2 \ll 1$. For these parabolic cases, by substituting {4.2} into the self-focusing equation {3.14}, the following result is obtained

$$\partial_{zz} f(z) = \frac{1}{k^2 a^2 f^3}\left(\frac{1}{a^2} - \frac{1}{a_E^2}\right) \tag{4.20}$$

Thus, if $a^2 > a_E^2$, then $\partial_z f < 0$ and self-focusing will occur (i.e. $f(z)$ will decrease). Self-focusing will not occur if $a^2 < a_E^2$ since $\partial_z f > 0$ and the beam will diffract continuously with f increasing with z. So a_E is the critical beam size for which self-focusing will occur and for a given intensity, a_E determines the critical power, that above which, self-focusing will occur. For $P < P_{cr}$, no self-focusing will take place.

We now investigate nonequilibrium solutions for the

beam size parameter f by introducing the diffraction distance $R_d = ka^2$, which is a familiar quantity in linear optics. Its significance can be seen by setting the focusing term in {3.14} to zero (i.e. $E_0^2 \rightarrow 0$) so that it becomes

$$\partial_{zz} f(z) = \frac{1}{R_d^2 f^3}$$

with solution

$$f^2 = 1 + z^2 / R_d^2 \tag{4.21}$$

where the boundary conditions $f(0)=1$ and $\partial_z f(0)=0$ have been used. Thus a beam propagating a distance $z=R$ will have increased its beam size by a factor of $\sqrt{2}$ due to diffraction. We can introduce a similar quatity $R_s = kaa_E$, which we will call the self-focusing distance, and it represents the distance the beam has to propagate for the width to decrease by a factor of $\sqrt{2}$, in the absence of diffraction. In terms of R_d and R_s we can write {4.20} as

$$\partial_{zz} f(z) = \frac{1}{f^3} \left(\frac{1}{R_d^2} - \frac{1}{R_s^2} \right) \tag{4.22}$$

Thus self-focusing will only occur if $R_s < R_d$, (i.e. if focusing dominates over diffraction). We recall that {4.22}

will only be valid for the cases where Φ is parabolic in E_0.

The solution of {4.22} with f(0)=1 and $\partial_z f(0)=0$ is

$$\{4.23\} \qquad f^2 = 1 + \left(\frac{1}{R_d^2} - \frac{1}{R_s^2} \right) z^2$$

so that for $R_s < R_d$, the beam will converge slowly.

For the ponderomotive nonlinearity in a strong magnetic field, we have from {4.7}

$$\{4.24\} \qquad R_s = \frac{2kc^2a^2}{\omega_{p-}^2 \beta_0 \alpha_r E_0^2} , \qquad \beta_0 << 1$$

and for the weak heating case

$$\{4.25\} \qquad R_s = \frac{2kc^2a^2}{\omega_{p-}^2 \alpha_h E_0^2} , \qquad \alpha_h E_0^2 << 1$$

For a CO_2 laser heated plasma with, $\omega_p^2/\omega^2 \sim 0.1$, $T_e \sim 100$ eV and $\beta_0 \sim 0.1$, {4.24} and {4.25} say that a self-focusing distance of 0.1 cm requires intensities of 10^{14} W/cm^2 and 10^{10} W/cm^2 respectively, where again we see that the ponderomotive nonlinearity requires much higher intensities than the weak heating case to give the same effect on focusing.

In Figure 3 we plot {4.23} for various values of

R_d/R_s . It can be seen how the beam converges more rapidly as the self-focusing distance decreases. It must be noted that {4.23} is only valid for $\alpha_r\beta_o E_o^2/f^2 \ll 1$ or $\alpha_h E_o^2/f^2 \ll 1$ so that as f decreases the approximations will be in jepardy. Thus the prediction of catastrophic self-focusing (i.e. $f \rightarrow 0$ as $z \rightarrow R_d R_s/(R_d^2-R_s^2)$) by {4.23} is invalid, and we expect new features to appear when f gets smaller.

4.3 NONEQUILIBRIUM SELF-FOCUSING - OSCILLATORY WAVEGUIDE STRUCTURES

In this section we examine solutions of the self-focusing equation for which the non-parabolic form for Φ is used. This corresponds to situations where the initial intensity is sufficiently high or where focusing has caused the intensity to become very large such that the approximations used in the previous section break down. The ponderomotive mechanism is not treated in this section as no analytic solutions can be obtained for $\alpha_r E_o^2 \geq 1$.

We begin by considering the weak heating case and rewrite equation {3.14} by defining a new function $U_1(f)$,

$$U_1(f) = \frac{-1}{2R_d^2 f^2} + \frac{\omega^2 a^2}{2c^2 R_d^2}\,\Phi\!\left(\frac{E_o^2}{f^2}\right) \tag{4.26}$$

where Φ is given by {2.58}. Then the self-focusing equation {3.14} can be written as

{4.27} $$\partial_{zz} f(z) = \partial_f U_1(f)$$

Equation {4.27} yields, upon integration, the conservation law

{4.28} $$\frac{1}{2}(\partial_z f)^2 - U_1(f) = C$$

where C is a constant. Using the boundary conditions at z=0

{4.29} $$f(0) = 1$$
$$\beta(0) = f^{-1}(0)\,\partial_z f(0) = \frac{1}{R_0}$$

where the last condition is derived from the fact that $\beta(0)$ is identified as the initial radius of curvature of the incident beam. If the incident beam is a finite plane wave, then $\partial_z f(0)=0 \Rightarrow R_0 \rightarrow \infty$. Thus, evaluating {4.28} at z=0 with the boundary conditions {4.29} determines the value of C as

{4.30} $$C = \frac{1}{2R_0^2} - U_1(1)$$

We can now make use of {4.28} to compute the minimum

value attained by the beam size parameter f. The existence of f_{min} comes from the fact that as the beam focuses, the effective intensity of the beam increases, causing the second term on the right-hand-side of {3.14} (the focusing term) to decrease in magnitude. This decrease in magnitude is due to the fact that Φ' is essentially the derivative of the density ratio η, and as the beam focuses, the increased intensity causes more and more density depletion in the beam; thus $\eta \rightarrow 0$ and $\eta' \rightarrow 0$ as $f \rightarrow 0$. Therefore the nonlinear self-focusing saturates at high intensities. At the same time as the nonlinearity weakens, the diffractive 'force' (the $R_d^{-2}f^3$ term in {3.14}) is increasing. Thus at some point the diffraction can dominate the focusing and the beam will eventually start to diverge, even though initially the beam was converging.

Using {4.28}, together with the fact that $\partial_z f=0$ at $f=f_{min}$ and taking $R_0 \rightarrow \infty$, we find that f_{min} satisfies

$$U_1(f_{min}) = U_1(1) \qquad \{4.31\}$$

We have solved {4.31} for f_{min} as a function of $\alpha_n E_0^2$; these results appear in Figure 4 and show the saturating behaviour of the focusing. Similar results have recently been published by Sodha et al (1976).

The strong heating case (2.3) will now be considered in much the same way. We begin by writing the self-focusing

equation {3.15} explicitly as

{4.32} $$\partial_{zz} f(z) = \frac{1}{k^2 a^2 f^3}\left[\frac{1}{a^2} - \frac{\omega_{p}^2}{c^2}\,\frac{f^2 \alpha_s E_0}{(1+2\alpha_s E_0)^{1/2}(1+\alpha_s E_0)}\right]$$

By defining a function $U_z(f)$ as

{4.33} $$U_z(f) = \frac{-1}{2k^2 a^4 f^2} - \frac{\omega_{p}^2}{k^2 c^2 a^2}\,\frac{\alpha_s E_0 \ln f}{(1+2\alpha_s E_0)^{1/2}(1+\alpha_s E_0)}$$

then {4.32} becomes

{4.34} $$\partial_{zz} f(z) = \partial_f U_z(f)$$

as we had before. Thus to compute f_{min}, we have to solve

{4.35} $$U_z(f_{min}) = U_z(1)$$

where we have used the boundary conditions $\partial_z f(0)=0$ and $f(0)=1$. The solution of {4.35} as a function of $\alpha_s E_0$ appears in Figure 5 where again we see the saturation behaviour.

5. SOME PHYSICS AND SOME APPLICATIONS OF SELF-FOCUSING

In this chapter we discuss the previous analytical results in terms of some simple physical arguements and draw some conclusions based on a physical understanding of self-focusing. We also discuss some possible applications of these ideas to the laser - solenoid fusion concept.

5.1 PHYSICAL DISCUSION OF SELF-FOCUSING

The self-focusing process comes down to a simple competition between two effects; the tendency of the beam to focus due to the density depletion in the beam caused by radiation pressure or heating, and the tendency of the beam to defocus due to diffraction. The dominant of the two effects determines the nature of the beam propagation. A balance can also occur between the two effects, as was seen in 4.1, where the diffraction of the beam was exactly compensated for by the focusing action of the plasma resulting in the propagation of a uniform beam. For a given intensity, this balance condition allowed us to determine the beam size and hence the power for which this self-trapped or equilibrium propagation state would exist. In 4.2 we saw that for a given intensity, focusing would only occur if the beam size 'a' was larger than the equilibrium value a_E. This is because a larger beam size reduces the

diffractive 'force' and the focusing action is able to dominate, causing the beam to converge.

According to the weak focusing solutions of 4.2, once the focusing began it was irreversible and catastrophic, i.e. the beam would shrink to zero at a point focus, causing the intensity to become infinite, which is an unphysical, if not uncomfortable conclusion. However, as the intensity increases, the validity of the approximations used in 4.2 is destroyed and a more rigorous discussion must be used. In 4.3 we found that the beam size reached a non-zero minimum value, which we can expect for two reasons. First, it is well known in optics that the focal spot size of any optical system is limited by diffraction and this will prevent the beam from focusing to a point. Second, the nonlinearity saturates at high intensity causing the strength of the focusing term to decrease to zero. This can be seen in the sketch (Figure 6) of the nonlinear term Φ of the dielectric. As the focusing depends on the derivative of Φ with respect to the electric field, then the saturation of the nonlinearity at high intensities causes $\Phi' \rightarrow 0$ thus weakening the focusing term. At the same time the diffraction effect is getting stronger since the beam is getting smaller. Thus, even though the focusing effect may indeed dominate initially, the subsequent convergence of the beam causes the nonlinearity to eventually weaken (saturate) and diffraction to increase to the point where it dominates, turning the focusing around and thereby spreading the beam.

However, as the beam spreads, the diffraction effect begins to weaken and we can eventually return to the initial situation where the focusing will again dominate over diffraction. Thus an oscillatory behaviour ensues comprised of alternate focusing and defocusing. This is depicted in Figure 7.

The effects of diffraction and saturation of the nonlinearity can also explain the existence of the minimum equilibrium beam size $a_{E)min}$, calculated in 4.1. The typical size for $a_{E)min}$ was $\sim c/\omega_p$ for all mechanisms. This size is (for $\omega_p \lesssim \omega$) of the order of the free space wavelength of the radiation and is a general feature of saturable nonlinear media (Akmanov <u>et</u> <u>al</u> (1966)). This represents the diffraction limited beam size for if $a<a_{E)min}$, diffraction is so strong that the nonlinearity cannot balance diffraction so that an equilibrium condition cannot be attained. Even if the intensity is increased the nonlinearity weakens (saturates) and is less effective in compensating for diffraction.

5.2 EFFECTS OF ABSORPTION ON SELF-FOCUSING

In the previous analysis of self-focusing (Chapter 4) we had assumed that absorption of the laser could be ignored which meant that the absorption length, l_{abs}, was much greater than lengths over which the intensity varied due to

focusing (or defocusing). For example if $l_{abs} \simeq R_d$, then we would expect the effect of absorption to be significant. As the absorption length varies as $T_e^{3/2}/n_e^2$ (for classical inverse-bremsstrahlung) we should solve for self-focusing together with absorption in a self-consistent manner i.e. any change in intensity brought about by absorption will change the plasma parameters which in turn affect the absorption. However, this is extremely difficult analytically so we will satisfy ourselves by examining a simplified picture and look for gross features.

We take the absorption coefficient to be a constant and replace the intensity E_o^2 by $E_o^2 e^{-k_a z}$ where k_a is constant. The weak focusing limit, 4.2 is treated in this way so that the self-focusing equation {4.22} becomes

$$\partial_{zz} f(z) = \frac{1}{f^3}\left(\frac{1}{R_d^2} - \frac{e^{-k_a z}}{R_s^2} \right) \tag{5.1}$$

The solution to {5.1} with no absorption (i.e. k_a=0) is

$$f^2 = 1 + \left(1 - R_d^2/R_s^2\right) z^2/R_d^2 \tag{5.2}$$

from which we can define a focal length z_f (i.e. when f=0) of,

$$z_f^2 = \frac{R_d^2 R_s^2}{R_d^2 - R_s^2} \tag{5.3}$$

Now z_f is an order of magnitude estimate for the scale

length of axial variation of the beam so that the neglect of absorption requires the condition

{5.4} $$z_f^2 << l_{abs}^2$$

If {5.4} holds, then for distances much smaller than the absorption length, $e^{-k_a z} \simeq 1$ and {5.1} becomes the equation previously solved in 4.2 where absorption was neglected. However, if the absorption length is very short, then the magnitude of the self-focusing term in {5.1} decreases rapidly with z and if the absorption length is sufficiently short such that

{5.5} $$z_f^2 >> l_{abs}^2$$

then for distances larger than a few absorption lengths, {5.1} becomes

{5.6} $$\partial_{zz} f(z) \simeq \frac{1}{R_d^2 f^3}$$

with solution

{5.7} $$f^2 = 1 + z^2/R_d^2$$

Thus if the absorption is sufficiently strong, it can cause the beam propagation to become diffraction dominated and the beam will thus spread, even though initially it may be

focusing. Thus, absorption is a defocusing mechanism which is to be expected as it weakens the nonlinearity by decreasing the laser intensity. Sodha et al (1976) have recently solved {5.1} numerically and have demonstrated the defocusing nature of absorption.

In Figure 8 we have plotted the weak (equation {5.2}) and strong (equation {5.9}) absorption solutions of {5.1}. For intermediate values of the absorption length, solutions to {5.1} must lie in the region bounded by the weak and strong absorption solutions as is clearly seen in the numerical results of Sodha et al . We also note that even if the absorption is weak, all beams will eventually defocus once the propagation distance has become comparable to the absorption length so that self-focusing can only be maintained for distances of the order of $\lesssim \ell_{abs}$.

5.3 SCALE LENGTHS OF SELF-FOCUSING

In this section we turn our attention to the size of scale lengths over which quatities of interest vary, namely the beam size and focusing distance (or oscillation wavelength). The latter quantity refers to the axial scale length of amplitude variation brought about by the focusing and defocusing of the beam while the former is a radial scale length. In particular we wish to examine the periodic propagation structure found in the discussion of the weak

and strong heating cases. As the temperature and density are functions of the electric field, then the alternate focusing and defocusing of the beam in the plasma will cause the temperature and density to vary periodically over the same scale length (which we call λ_{osc}) creating an axial nonuniform plasma. We wish to estimate this scale length to see what axial variation to expect.

We recall the results of the weak focusing solutions of section 4.2 where we could define a focal distance z_f to be

$$z_f^2 = \frac{R_s^2 R_d^2}{R_d^2 - R_s^2} \tag{5.8}$$

which for $R_s \simeq R_d$, could be quite long. However, we are interested in solutions for higher intensities (i.e. $\alpha_h E_o^2$ or $\alpha_s E_o$ of $O(1)$). For these cases we can write the self-focusing equations {3.14} and {3.15} as

$$\partial_{zz} f(z) = \frac{1}{f^3}\left(\frac{1}{R_d^2} - \frac{1}{R_s^2(f)} \right) \tag{5.9}$$

where

$$\{5.10\} \quad R_s^2(f) = \begin{cases} \dfrac{R_d^2}{\xi^2} \dfrac{(1+\alpha_h E_o^2/2f^2)^2}{\alpha_h E_o^2/2} \\ \\ \dfrac{R_d^2}{\xi^2} \dfrac{(1+2\alpha_s E_o)^{1/2}(1+\alpha_s E_o)}{f^2 \alpha_s E_o} \end{cases}$$

where $\xi^2=\omega_p^2a^2/c^2$. If we take $R_s(f)$ in {5.10} to be given by $R_s(1)$ (i.e. $f=1$) and treat this as a constant, then {5.9} has the simple solution obtained in {4.1} of

$$\{5.11\} \qquad f^2 = 1 + \left[1 - \frac{R_d^2}{R_s^2(1)}\right]\frac{z^2}{R_d^2}$$

and setting $f=0$ yields, in analogy with {5.8}, a 'focal' distance z_f' of

$$\{5.12\} \qquad z_f'^2 = \frac{R_d^2 R_s(1)}{R_d^2 - R_s^2(1)}$$

Equation {5.12} gives an estimate , albeit a crude one, for the scale length of axial variation of the electric field. The oscillation length would be $\sim 2z_f'$. For $\alpha_h E_o^2$ or $\alpha_s E_o$ of $O(1)$, {5.12} yields

{5.13} $$z_f'^2 \simeq \frac{R_d^2}{f^2} = \frac{c^2 k^2 a^2}{\omega_p}$$

Although the derivation of {5.13} is not a very exact one, it reveals an important property of the focal or oscillation length in that this length is of the order of the initial beam size 'a' for the higher intensities. This can be seen more rigorously if, following Max (1976) we consider a beam propagating in the equilibrium self-trapped state and perturb the solution slightly. This is done by writing the right-hand side of the self-focusing equation as a function $\psi(a^2,f)$, and then expanding ψ about the equilibrium in order to linearize the equation.

We treat the weak heating case here, as the strong heating case gives very similar results. Following the above procedure, equation {3.14} becomes

{5.14} $$\partial_{zz} f(z) = \psi(a^2, f(z))$$

where

{5.15} $$\psi(a^2,f) = \frac{1}{k^2 a^2 f^3}\left(\frac{1}{a^2} - \frac{\omega_p^2 \alpha_h E_0^2/2}{(1+\alpha_h E_0^2/2f^2)^2}\right)$$

We expand ψ as

{5.16} $$\psi(a^2,f) = \psi(a_E^2,1) + (f-1)\partial_f \psi(a^2,1) + (a^2-a_E^2)\,\partial_{a^2}\psi(a_E^2,f) + \cdots$$

Then using {5.16}, {5.14} can be written as

{5.17} $$\partial_{zz}(f-1) + (f-1)\frac{2}{k^2 a_E^4}\frac{\alpha_h E_o^2}{\left(1+\frac{\alpha_h E_o^2}{2}\right)} \simeq 2\,\frac{a_E^2 - a^2}{k\, a_E^6}$$

which has the solution that satisfies f=1 and $\partial_z f=0$ at z=0,

{5.18} $$f-1 = \frac{2(a_E^2-a^2)}{k^2 a^2 a_E^2}\left[1-\cos\left[\frac{1}{k a_E^2}\left(\frac{2\alpha_h E_o^2}{1+\alpha_h E_o^2/2}\right)^{1/2} z\right]\right]$$

Thus the perturbed solutions oscillate with Λ_{osc} given by

{5.19} $$\Lambda_{osc} = \sqrt{2}\,\pi\, k a_E^2\left(\frac{1+\alpha_h E_o^2/2}{\alpha_h E_o^2}\right)^{1/2}$$

As the initial beam size was $\sim a_E$, then we see again that the oscillation scale length is determined, in part, by the beam size. This feature also appears in treatments of beam-trapping where the density profile is assumed to be given, instead of solving for it self-consistently (see Mani <u>et al</u> (1974), Feit and Fleck (1976)), where it is found that Λ_{osc} is determined by the width of the density profile. As the beam size determines the width of the density profile (in the steady state) then this is the reason for the connection between beam size and Λ_{osc}.

It is worthy to note that {5.13} or {5.19} predict that for beam sizes of the order of the radiation wavelength λ, then $\lambda_{osc} \sim O(\lambda)$, which violates the assumption that the axial variation of the amplitude is slowly varying (see discussion after {3.5}) which helped to simplify the wave equation. Thus for the previous discussion to be valid, we require the initial beam size 'a' to be much larger than the radiation wavelength, i.e. $a \gg k^{-1}$. Therefore, beams that are of the order of $a_E)_{min} \sim c/\omega_p$ tend to violate our slowly varying assumption, so that for $a \gg a_E)_{min}$, our solutions should have $\lambda_{osc} \gg k^{-1}$, even for high intensities. This is in contrast with the conclusion of Max (1976) whereby she claims that $\lambda_{osc} \simeq k^{-1}$ for high intensities, regardless of initial beam size.

5.4 SMOOTHING OF INHOMOGENITIES

Mani et al (1974) pointed out that the axial variations of temperature and density that accompany the periodic self-focusing will be smoothed out by axial thermal conduction. The conduction along magnetic field lines is due primarily to electrons and lengths over which conduction keeps the temperature uniform is given by

{5.20} $$l_{\parallel} = \frac{3.5 \times 10^{9}\, T_e^{5/4}\, \tau^{1/2}}{n_e^{1/2}}$$

where τ is the time scale involved. For a 1 keV plasma with n_e=7x10^{17} cm^{-3}, $l_{\parallel} \sim 1$ cm for $\tau \sim 10$ nsec. As the heating time scales of interest in fusion reactors are nearing 1μsec, then the temperatures and hence density should be fairly uniform over distances of tens of centimeters. In constrast, λ_{osc} is of the order of a few beam sizes, and typically a$\sim$0.1 - 1.0 cm, so we see that $\lambda_{osc} \ll l_{\parallel}$ at very high temperatures and thus the inhomogeneity in density and temperature introduced by the non-uniform intensity should be smoothed out.

In the radial direction, perpendicular to the magnetic field, the density and temperature variations are determined by the beam width, and as the beam focuses, the radial width of the density and temperature profiles also decrease. However, radial thermal conduction will limit the size of these profiles. Heat conduction across field lines is primarily due to ions because of their larger Larmor radii, and the length over which radial heat conduction will keep the temperature and density uniform is given by

{5.21} $$l_{\perp} = 10^{-1}\, n_e^{1/2}\, \tau^{1/2}$$

For $B_o \sim 100$ kG, $n_e \sim 7\times10^{17}$ cm^{-3}, $T_e \sim 1$ keV then {5.21} gives $l_{\perp} \sim 10\mu$ and $10^3\mu$ for $\tau \sim 1$ns and 1μs respectively. So for CO_2 radiation ($k^{-1} \sim 10\mu$), even for short time scales, it is unlikely that beam sizes of $a \sim k^{-1}$ would self-focus as a density profile that narrow would not be maintained

sufficiently long before conduction smoothed it out. Thus for a longer time scale, the perpendicular heat conduction can cause the temperature and density profiles to be fairly uniform across the extent of the beam, which will tend to weaken the focusing effect, causing the periodic focusing to smooth out. In fact, if the radial thermal conduction is very 'efficient', then this could untrap the beam by not permitting a density profile to be maintained. Thus in order to cut down on radial thermal conduction, a strong magnetic field inside the plasma will probably have to be used in order to trap a laser beam over large distances, as required for laser - solenoid fusion schemes.

5.5 APLICATIONS TO LASER SOLENOID FUSION

We have seen in our discussion of the ponderomotive and heating mechanisms that the latter is by far the more dominant self-focusing process for intensities of interest in laser - solenoid fusion (10^{11} - 10^{12} W/cm^2), even in strong magnetic fields. However, the focusing of the laser beam due to heating and subsequent expansion of the plasma causes the intensity to increase by factors as high as 10^2 - 10^3 so that the ponderomotive force may become important in these high intensity regions either by causing further plasma depletion or by coupling the transverse e.m. radiation to longitudinal plasma modes (i.e. stimulated scattering processes). Nevertheless we have seen that the

heating of the plasma can create a density profile that is 'favourable' to affect focusing even though the plasma was initially uniform.

In the foregoing analysis, we have computed the density profiles self-consistently, but these are only valid when a steady-state condition has been reached. This means that these profiles are established on time scales of the order of the characteristic heating or phonon transit times (τ_h or τ_{ac}, see introduction). Thus we have not solved the self-focusing problem for time scales shorter than these, leaving the question of the focusing (or defocusing) of the initial portion of the beam unanswered. We can, however glean some qualitative information on the development of the initial portion of a laser pulse by considering a slowly increasing pulse. If the rate of increase of the pulse is sufficiently slow so that we can apply our weak heating theory as an approximation over time scales $\Delta\tau \ll \tau_{laser}$, then from the self-focusing equation {3.14}, we see that for early times in the pulse, where the intensity is very small, the focusing term can be neglected and diffraction initially dominates. For later times the diffraction should become compensated for by the focusing action as the pulse intensity begins to increase and at some point the self-trapped state should be attained. As the pulse intensity increases further, the periodic focusing and defocusing discussed in 3.3 will ensue. If we recall that absorption is a defocusing mechanism and that for early times the

plasma will be relatively cold implying short absorption lengths, then this process will tend to cause the early portions of the pulse to defocus as well as restrict any focusing that may take place, to regions of length less than l_{abs}. These qualitative features appear in Figure 9.

The qualitative conclusions, as shown in Figure 9, have been born out in a recent time dependent numerical treatment of the plasma radial dynamics and axial beam propagation by Feit and Fleck (1976), where they have seen features similar to Figure 9, even for short pulses.

From the above discussion, we can conclude that the initial portion of the laser pulse will diffract out of the plasma as sufficient time has not elapsed to establish a density profile sufficient to affect beam-trapping. Thus, unless there exists a pre-formed, favourable density profile, the beam front will tend to defocus. This will probably also be the tendency of the beam just behind the bleaching wave front (the sharp boundary between optically thick and thin regions of the plasma column, see Steinhauer and Ahlstrom (1975)). However, the beam may produce a density minimum ahead of the bleaching front because in the regions far behind the front the plasma is being heated and is expanding, thereby blowing the magnetic field imbedded in the plasma out as well. This will cause field lines further ahead to become distorted and take some plasma with them, thereby creating a density minimum ahead as well. Now the

disturbance of the magnetic field will propagate at the Alfvén velocity $v_A = B/\sqrt{4\pi n m_i}$ so that if the bleaching wave propagates at or slower than v_A then the heating and subsequent expansion of the plasma (with the magnetic field) will tend to create a favourable density profile ahead of the front and may tend to focus the beam (or at least weaken the diffraction tendency) (Burnett and Offenberger (1976)). However, this requires a closer look at the coupled radial and axial dynamics of the laser - plasma interaction.

By taking diffraction into account, we have seen in 3.3, that even though catastrophic self-focusing does not occur, the intensities attained in the focused portions of the beam can be very high indeed. This is quite undesirable as this puts the intensity into a regime where parametric instabilities such as stimulated Brillouin scattering (Drake et al (1974)) may be important. Also this strong focusing can create nonuniformities in plasma density and temperature as discussed earlier. To reduce the strong focusing, we could use very large B-fields so that a nearly constant radial density profile is maintained thereby weakening the focusing tendency. As we saw that the strong focusing will not take place until some heating has caused a density well to form, the application of strong B-fields would not be required until the plasma was 'hot'. However, near the beam front, as the plasma temperature is cooler, high magnetic fields would not allow an appreciable density profile to form so that beam trapping may not even occur.

Thus to control the focusing in some fashion, the magnetic field would have to be increased in the high temperature, heated region of the plasma column and not in the bleaching front region. So a pre-determined rate of increase of B along the axis could help to reduce the strong focusing in the hot regions by reducing the amount of plasma depletion in the beam. High magnetic fields would also be neccessary to prevent diffractive losses at the 'hot' end of the solenoid (the end where the laser is incident) as radial heat conduction can smooth out a favourable density profile when high temperatures are reached unless a high magnetic field is present to reduce this effect. But the use of high magnetic fields is undesireable from the point of view of obtaining high beta plasmas for greater efficiency in fusion reactors, so that this is one problem which still has to be resolved.

To avoid the use of high magnetic fields to control strong focusing tendencies, the laser energy could be distributed over many different laser lines so that the beam would be composed of radiation spread over many wavelengths. As the focusing distance z_f or oscillation wavelength λ_{osc} depend on wavelength (see equations {5.13} or {5.19}), then the use of a multimode laser would tend to smear the focusing out since some of the modes would be focusing while at the same time others would be defocusing (Offenberger (1977)). As opposed to the case where all the laser energy

is in a single monochromatic line, giving rise to high intensities, any particular mode of a multimode laser would have a somewhat lower intensity so that strong focusing would tend to be less severe in terms of the high intensities that it would produce. This is the result of only some of the laser energy being focused into a small region at any particular point. Also the smearing out of the focusing would tend to smooth out inhomogeneities in temperature and density caused by nonuniform heating.

We end this chapter by mentioning those areas related to the propagation of laser beams in long plasma columns that require further investigation. First of all the stability of the laser beam - plasma system requires investigation. Steinhauer (1976) has shown that the beam is stable to long wavelength, axial perturbations (long compared with the focusing length), however Feit and Maiden (1976) claim that short wavelength perturbations can cause exponential growth of the beam size. Both of these investigations are quite preliminary.

Also, scattering due to nonlinear processes such as stimulated Brillouin or Raman scattering (SBS or SRS) may become important if the beam focuses to high intensities. Sodha et al (1976) have shown that self-focusing in an unmagnetized, collisionless plasma can enhance SRS considerably.

Perhaps the most important piece of work that is

required, apart from an actual experiment, is a completely self-consistent solution of the coupled, time dependent, plasma - optical problem where both radial and axial plasma dynamics are taken into account. Also the propagation of multimode lasers in plasmas requires investigation.

6. CONCLUSION

In this thesis we have shown that the nature of the laser beam propagation depends on the competition between the refractive (focusing) effect of the plasma and the diffractive (defocusing) tendency of the beam. As a result of this competition, the beam will either defract, propagate uniformly, or alternately focus and defocus, with the focal size limited by diffraction. This behaviour can be produced by either a collisionless ponderomotive mechanism or by the heating of the plasma. We have studied the effect of an externally produced magnetic field inside the plasma and shown that more laser energy is required to produce the same effect on the focusing of the beam as compared to the field free case, as the plasma depletion is reduced by the B-field. It was also shown that regardless of the self-focusing mechanism, the smallest equilibrium beam size is of the same order, whether or not a magnetic field is present. Thus we conclude that it is diffraction, which is common to all cases considered, that limits the beam size. The heating mechanism was found to be a far more dominant process than the ponderomotive nonlinearity for laser powers of interest to solenoid fusion, although the ponderomotive force may become important in the focal regions of the beam.

We have speculated that strong absorption of the laser may tend to defocus the beam front for early pulse times

while strong radial conduction may flatten the density profile at late times (when the plasma is hot) and hence destroy the focusing mechanism. A pre-formed density minimum may be necessary to help focus the initial beam front while to counter the effects of the strong focusing that can occur later, we have suggested that strong magnetic fields might have to be used or that the laser energy be distributed over many laser lines (i.e. use of a multimode laser). However, high magnetic fields will probably be required at late times to reduce the strong radial thermal conduction which could destroy the focusing profile.

Table 1. Restrictions for Weak Heating Model

T_o (eV)	$E_o^2 <<$ (statvolt/cm)^2	$B_o >>$ (kG)
10	1.3×10^{8}	7.2
10^2	1.3×10^{9}	23
10^3	1.3×10^{10}	72
10^4	1.3×10^{11}	230

(Note: the restriction on the magnetic field for a given temperature can be expressed as $\beta=8\pi n_o T_o/B_o^2<<7.7\times10^{-18}n_o$, which gives $\beta<<1$ for $n_o=7\times10^{17}$ cm^{-3}.)

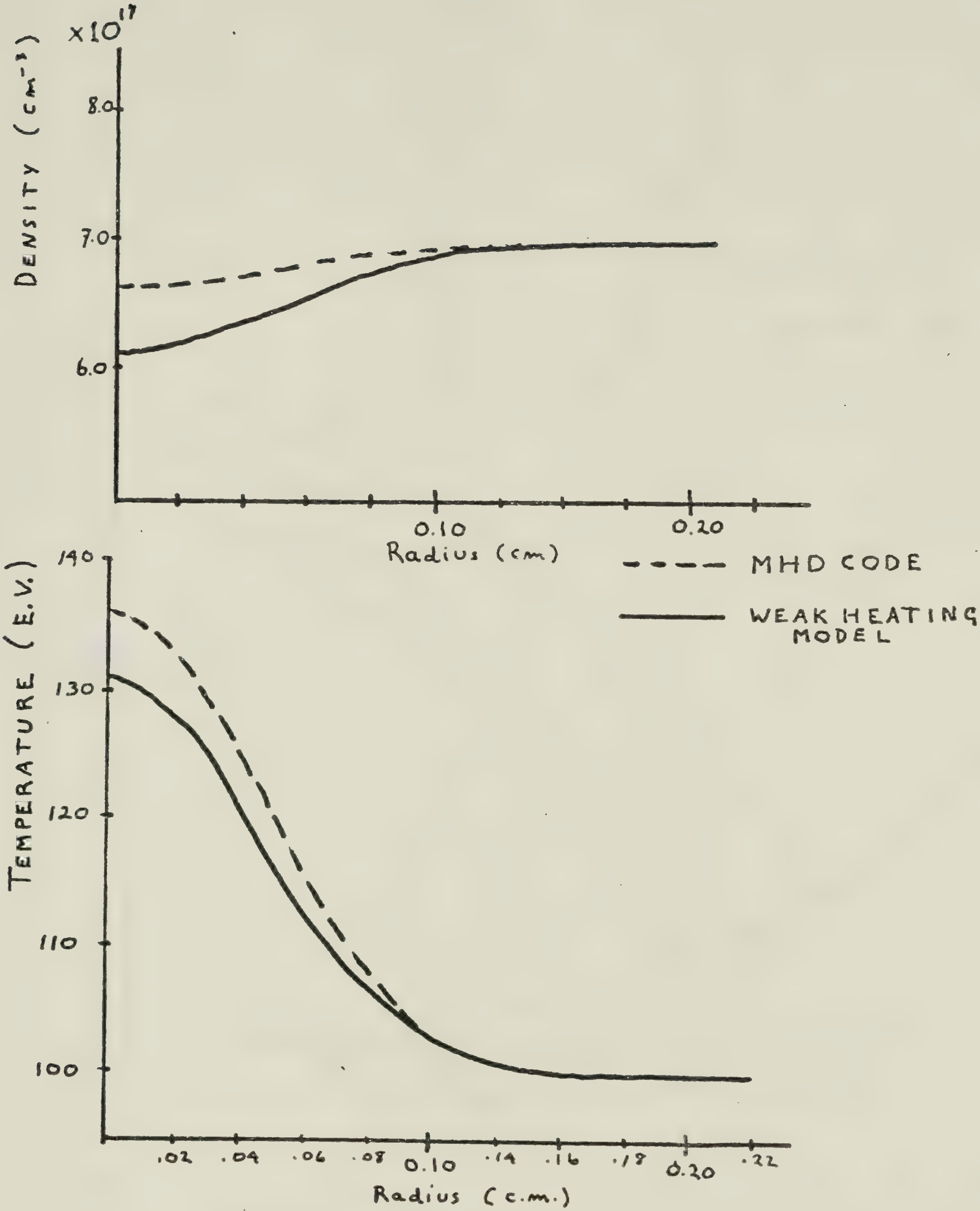

Fig. 1. Comparison of MHD code (– – – –) and Weak Heating model (———). $P=1.5\times10^8$ Watts, $T_0=100$ eV, $B_0=100$ kG, $n_0=7\times10^{17}$ cm^{-3}.

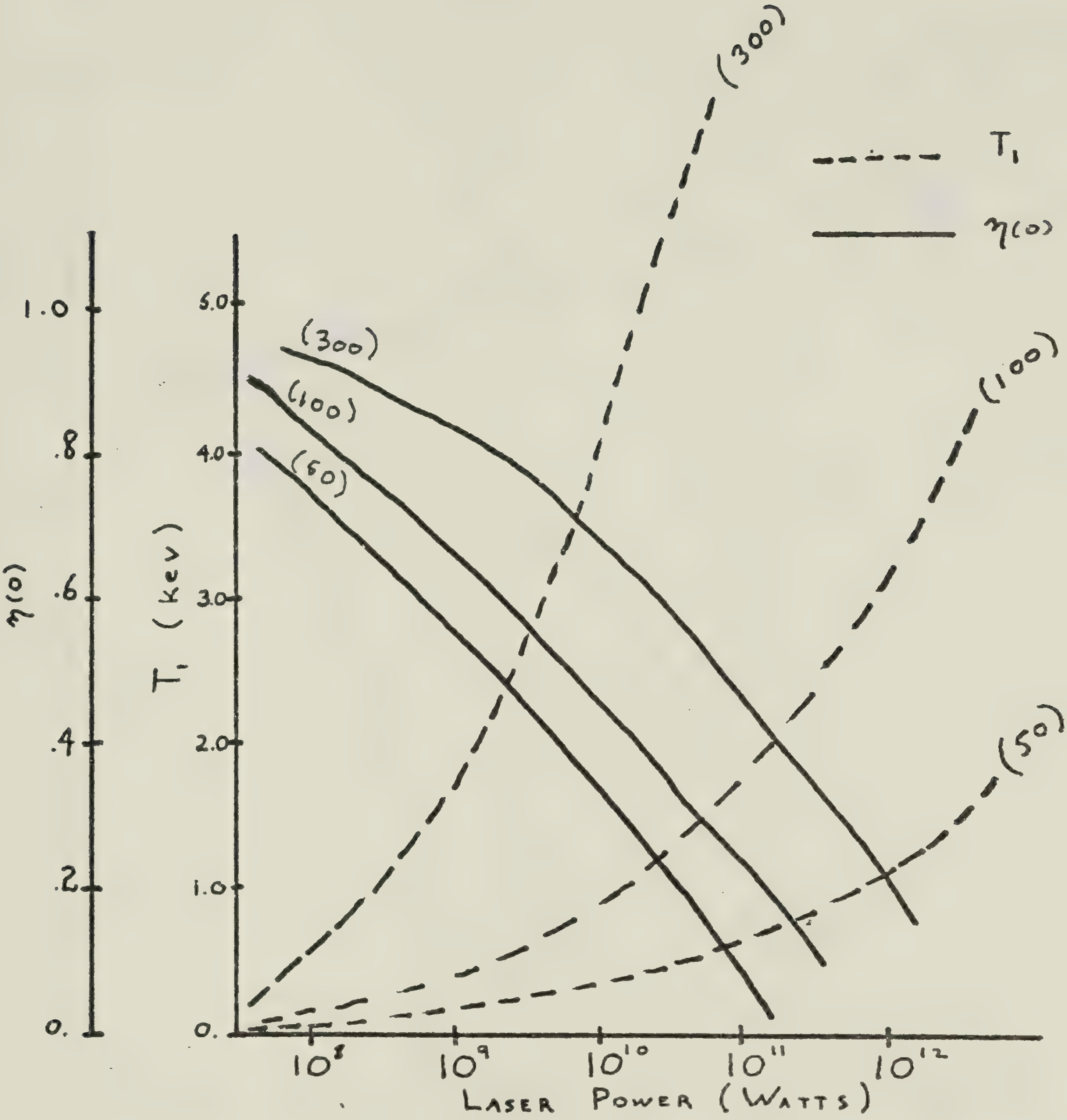

Fig. 2. On-axis temperature (T_1) and density ratio ($\eta(0)$) vs. $\alpha_s E_0$, from the Strong Heating model. The numbers in brackets are the external B-field (kG).

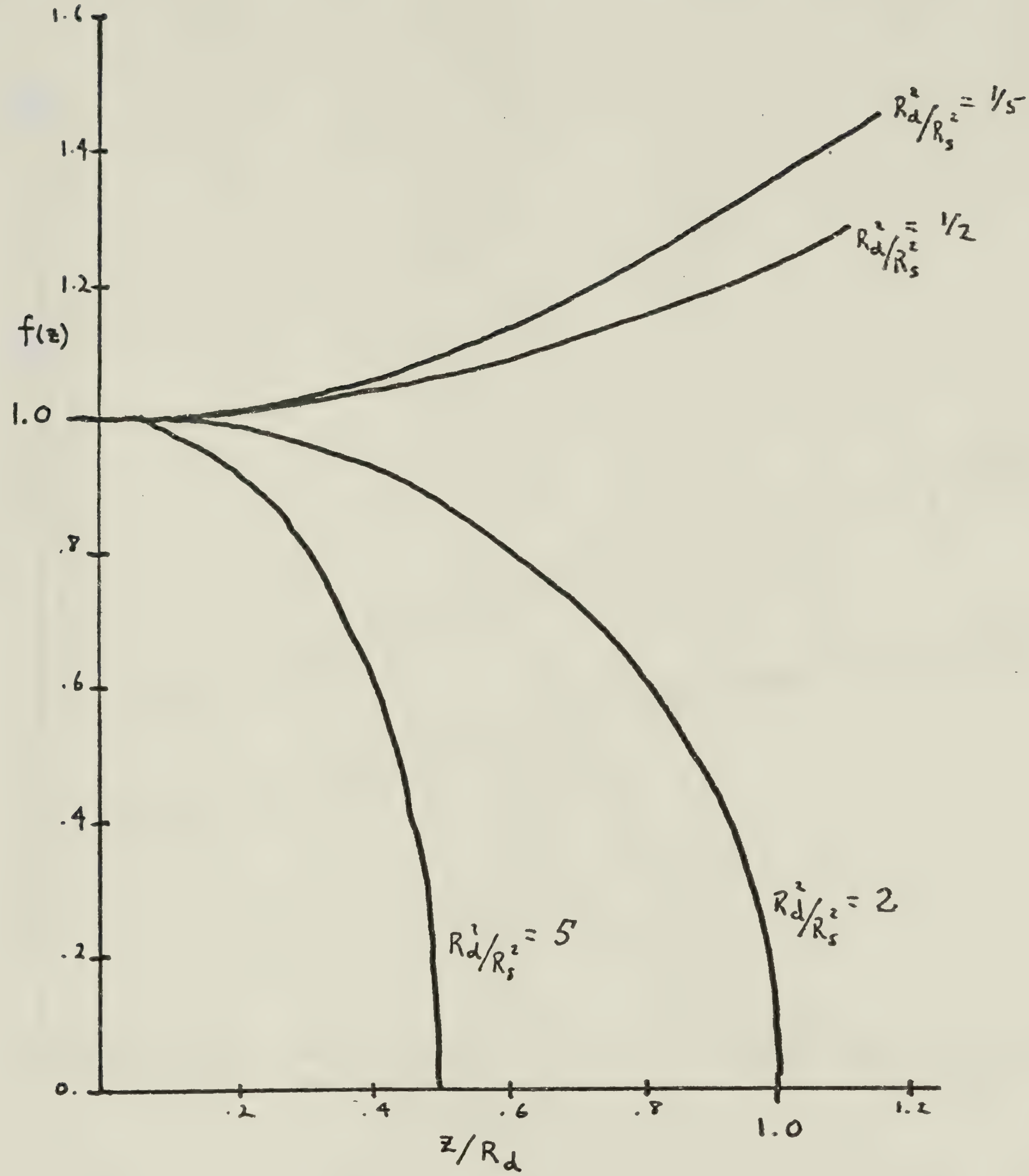

Fig. 3. Beam size parameter f(z) - Weak focusing limit. See equation {4.23}.

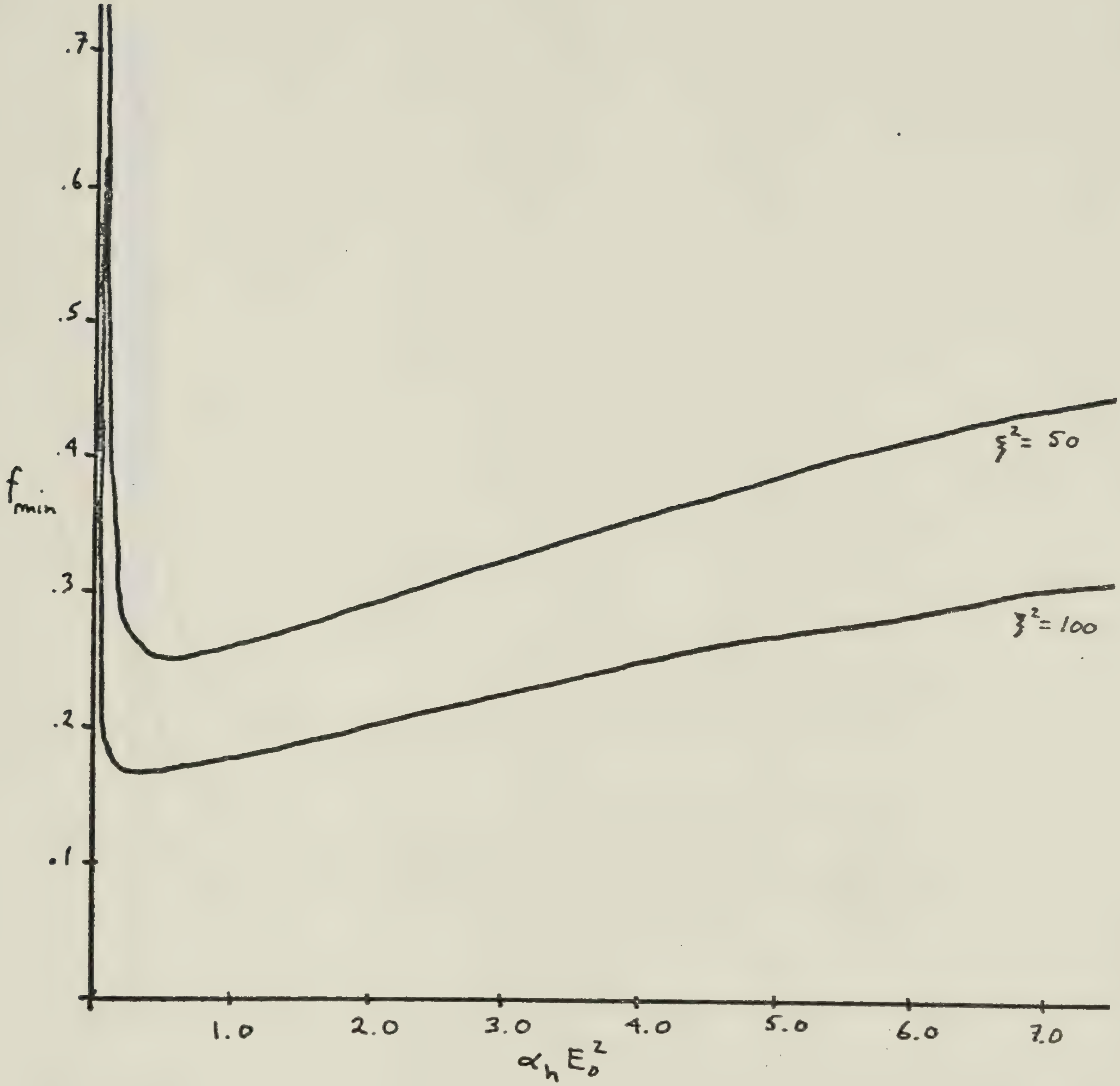

Fig. 4. Minimum beam size parameter f_{min} for Weak Heating model.

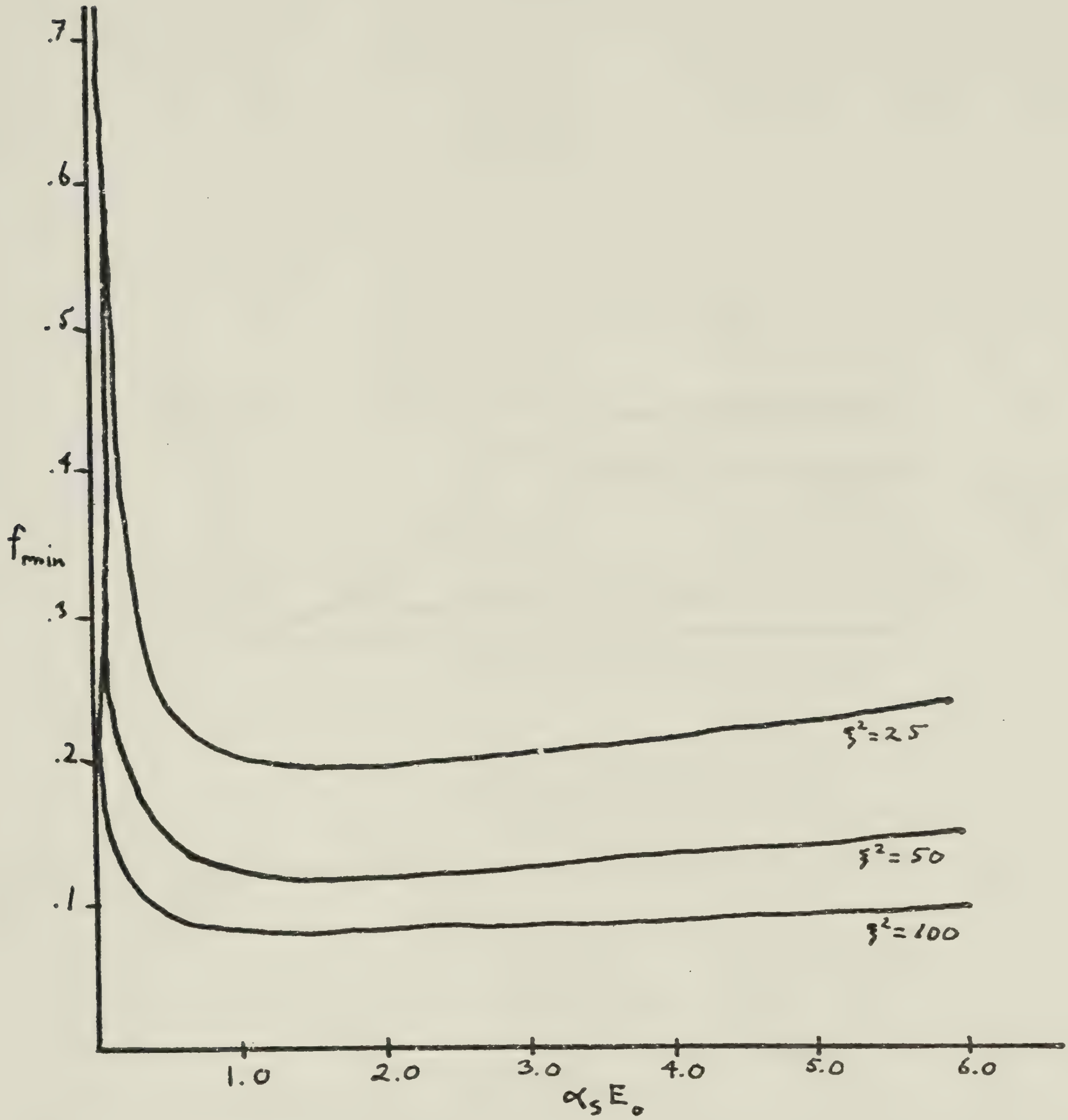

Fig. 5. Minimum beam size parameter f_{min} for Strong Heating model.

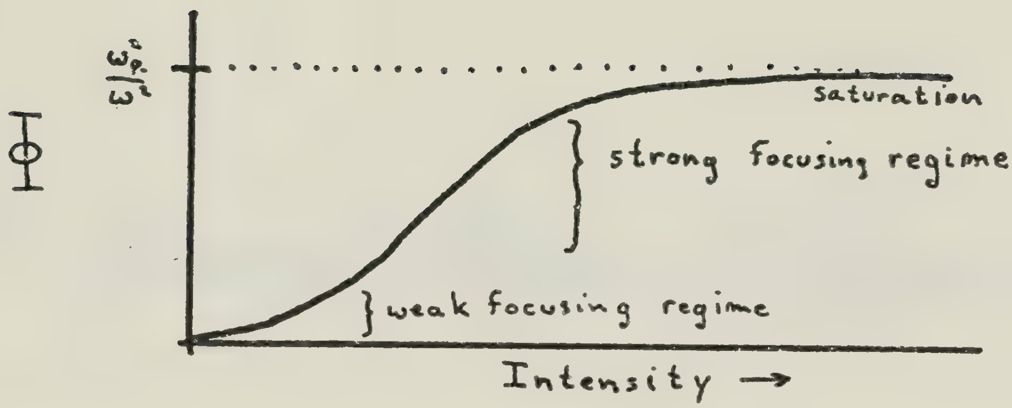

Fig. 6. Functional behaviour of the nonlinear part (Φ) of the dielectric ϵ.

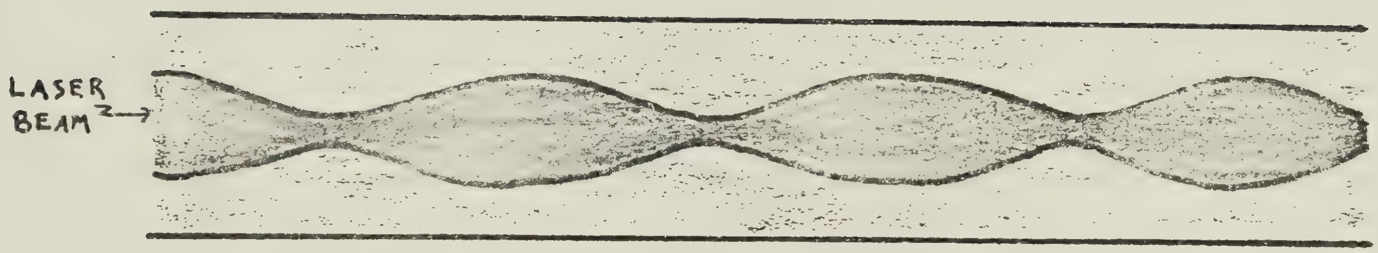

Fig. 7. Alternate focusing and defocusing of laser beam in a plasma.

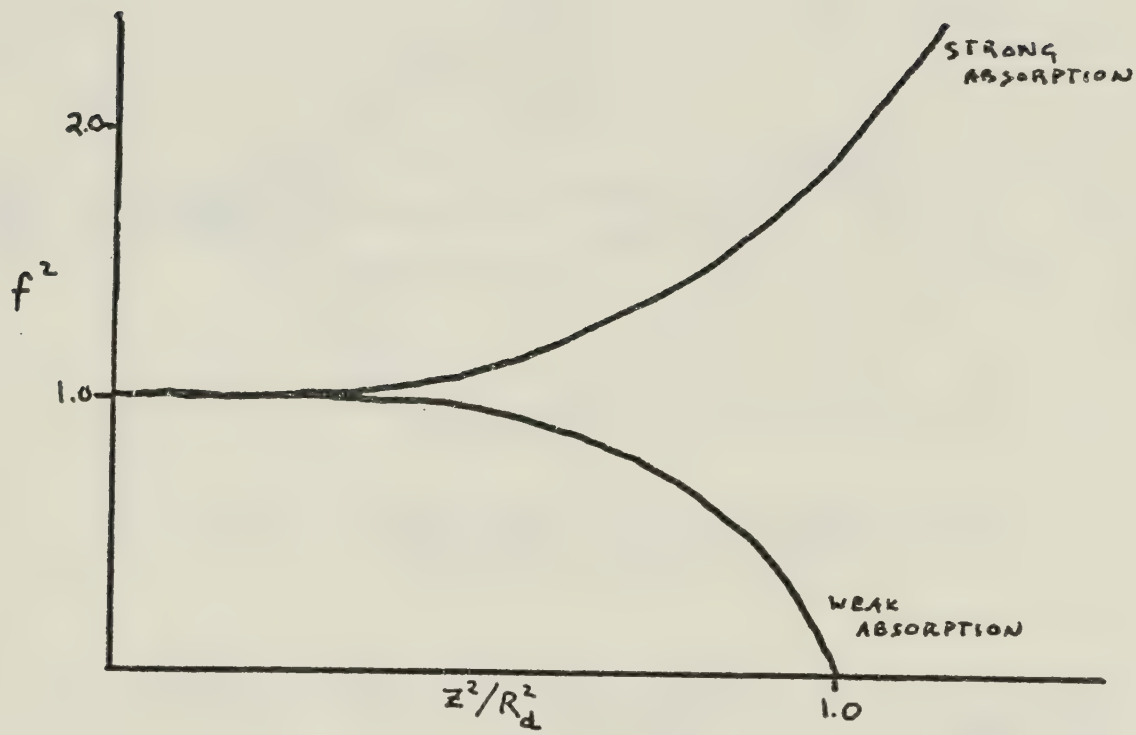

Fig. 8. Strong and weak absorption solutions. See equations {5.2} and {5.9}. $R_d^2=2R_s^2$.

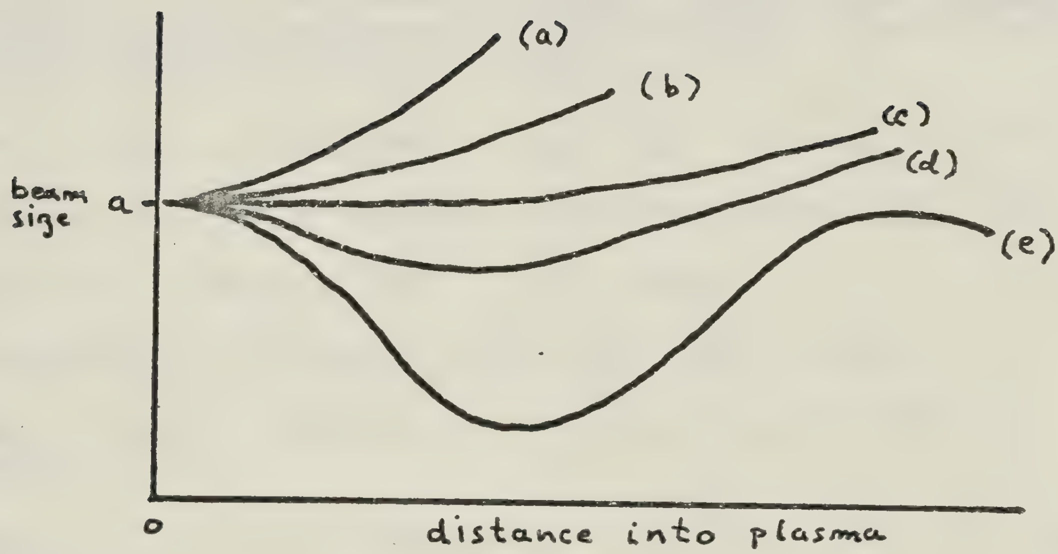

Fig. 9. Qualitative time developement of laser propagtion in a plasma. (a), and (b) correspond to early times in the pulse where diffraction dominates and spreads the beam although the spreading in (b) is somewhat diminished by focusing tendencies. (c) and (d) correspond to later times where focusing is balanced by diffraction ((c)) or dominates diffraction ((d)), although absorption may defocus the beam after propagating a short distance into the plasma. And (e) represents the late times when focusing completly dominates diffraction and the absorption lengths are very long.

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APPENDIX A DERIVATION OF EQUATION {2.45}

We derive the equilibrium equation for electrons by setting the d.c current to zero (i.e. $\vec{J}^0=0$). In an axial magnetic field there exists both a radial and θ-component of $\vec{J}^\circ$ so we set each component to zero to obtain the equilibrium equation. The same equation is obtained from both components so that we will treat only one, the -component here. The θ-component of the zero-order (d.c.) current is

$$\{A1\} \quad J_\theta^0 = \frac{4\pi}{3} e \int_0^\infty v^3 f_{1_\theta}' \, dv = -\frac{4\pi}{3} e \int_0^\infty \frac{\Omega_e}{\nu^2+\Omega_e^2} v^3 \left[v \partial_r f_0^\circ - \frac{e}{m_e} E^\circ \partial_v f_0^\circ \right] dv$$

The above equation is integrated using the frequency ordering $\Omega_e^2 >> \nu^2$ so that

$$J_\theta^\circ = -\frac{4\pi}{3} e \int_0^\infty \frac{v^3}{\Omega_e} \left[v \partial_r f_0^\circ - \frac{e E^\circ}{m_e} \partial_v f_0^\circ \right] dv$$

$$\{A2\} \quad = -\frac{4\pi}{3} e \left\{ \partial_r \left[\Omega_e^{-1} \int_0^\infty v^4 f_0^\circ dv \right] - \partial_r \Omega_e^{-1} \int_0^\infty v^4 f_0^\circ dv - \frac{e E^\circ}{m_e \Omega_e} \int_0^\infty v^3 \partial_v f_0^\circ dv \right\}$$

With $f_0^\circ = n_e A T_e^{-3/2} \exp(-av^2/T_e)$ from {2.42}, the integrations in {A2} are straight forward and we obtain

$$J_0^{\,o} = -\frac{4\pi}{3} e \left\{ \frac{3}{8} \frac{A}{a^{5/2}} \sqrt{\pi}\, \partial_r \left(\frac{n_e T_E}{\Omega_e} \right) + \frac{1}{\Omega_e^2} \partial_r \Omega_e \frac{3}{8} \frac{A \sqrt{\pi}\, n_e T_E}{a^{5/2}} + \right.$$

{A3} $$\left. + \frac{3}{4} \frac{e E^o}{m_e a^{7/2}} A \sqrt{\pi} \frac{n_e}{\Omega_e} \right\}$$

Setting J^o to zero yields the desired equation, {2.45}

{A4} $$\partial_r (n_e T_E) = -2 n_e e E^o$$

APPENDIX B DERIVATION OF EQUATION {2.78}

From {2.76} we have

$$\{B1\} \quad \frac{r^2}{\sigma_1^2 2} + \frac{\sigma_1^2 A(r)}{T_1^2} \exp\left(-r^2/2\sigma_0^2\right) - 2 = 0$$

Differentiating twice w.r.t. r yields

$$\{B2\} \quad \frac{1}{\sigma_1^2} + \frac{2\sigma_1^2 A(r)}{T_1^2}\left[\frac{1}{\sigma_1^2} - \frac{1}{2\sigma_0^2}\right]\exp\left(-r^2/2\sigma_0^2\right) +$$

$$+ \sigma_1^2 \frac{\partial_{rr} A(r)}{T_1^2} \exp\left(-r^2/2\sigma_0^2\right) + O(r^2) + \cdots = 0$$

Setting r=0 in {B2}, we obtain, using {2.73} and {2.77},

$$\{B3\} \quad \frac{1}{\sigma_1^2} + 4\left(\frac{1}{\sigma_1^2} - \frac{1}{2\sigma_0^2}\right) + \frac{2\,\partial_{rr}\eta(0)}{\eta^2(0)} = 0$$

Now

$$\{B4\} \quad \frac{\partial_{rr}\eta(0)}{\sigma_1^2\eta^2(0)} = \left[1 - \left(1 + \frac{\alpha_s^2 E_0^2}{1+2\alpha_s E_0}\right)^{-1/2}\right] \frac{\alpha_s E_0 (1+2\alpha_s E_0)^{1/2}}{\sigma_1^2}$$

so that for $\alpha_s E_0=1$, this term is $0.2/\sigma_1^2$, and it is safe to neglect the $\partial_{rr}\eta(0)$ term and we then obtain {2.78} from {B3} as

{B5} $$\sigma_1^2 \simeq \frac{5}{2}\sigma_0^2$$

If the last term in {B3} cannot be neglected, then its effect of the temperature profile width σ_1 would be to widen it. We can state that the validity of {B5} requires the use of sufficiently strong magnetic fields as $\alpha_s E_o \rightarrow \infty$ and $\eta(0) \rightarrow 0$ when $B_o \rightarrow 0$. If $\alpha_s E_o$ becomes too large the assumption of a Gaussian shaped temperature profile is probably in doubt as well.